Mathematical Finance

Exercise Sheet 10

Submit by 12:00 on Wednesday, December 6 via the course homepage.

Exercise 10.1 *(Construction of ζ)* Let $S = (S_t)_{0 \leqslant t \leqslant T}$ be an RCLL process with $S_0 = 0$.

(a) Assume S is locally bounded, so that there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times increasing stationarily to T with S^{τ_n} bounded for each n. Show that there exists a strictly positive predictable process $\zeta \in L(S)$ such the random variable

$$(\zeta \bullet S)^*_T := \sup_{0 \leqslant t \leqslant T} |\zeta \bullet S_t|$$

is bounded.

(b) Assume instead that S is a σ-martingale. Show that there exists a strictly positive predictable process $\zeta \in L(S)$ such the $(\zeta \bullet S)^*_T$ is integrable.

Exercise 10.2 *(Sum of σ-martingales is a σ-martingale)* Let S^1 and S^2 be σ-martingales. Show that the sum $S^1 + S^2$ is again a σ-martingale.

Exercise 10.3 *(Density of $\mathbb{P}_{\mathrm{e},\sigma}$ in $\mathbb{P}_{\mathrm{a},\sigma}$)* Let $S = (S_t)_{0 \leqslant t \leqslant T}$ be a P-semimartingale. Recall the set $\mathbb{P}_{\mathrm{a},\sigma}(S)$ defined by

$$\mathbb{P}_{\mathrm{a},\sigma}(S) := \{Q \ll P \text{ on } \mathcal{F}_T : S \text{ is a } Q\text{-}\sigma\text{-martingale}\}.$$

(a) Show that the sets $\mathbb{P}_{\mathrm{a},\sigma}(S)$ and $\mathbb{P}_{\mathrm{e},\sigma}(S)$ are convex.

(b) Assume that $\mathbb{P}_{\mathrm{e},\sigma}(S) \neq \varnothing$. Show that $\mathbb{P}_{\mathrm{e},\sigma}(S)$ is $L^1(P)$-dense in $\mathbb{P}_{\mathrm{a},\sigma}(S)$, in the sense that for each measure $Q \in \mathbb{P}_{\mathrm{a},\sigma}(S)$, there is a sequence $(Q^n)_{n \in \mathbb{N}} \subseteq \mathbb{P}_{\mathrm{e},\sigma}(S)$ such that $Z^n \to Z$ in $L^1(P)$, where Z^n and Z denote the densities of Q^n and Q with respect to P, respectively.