Mathematical Finance

Exercise Sheet 11

Submit by 12:00 on Wednesday, December 13 via the course homepage.

Exercise 11.1 *(Equivalence of (NA))* Show that S satisfies (NA) if and only if 0 is maximal in $\mathcal{G}_{\mathrm{adm}}$.

Solution 11.1 We have that S satisfies (NA) if and only if $\mathcal{G}_{\mathrm{adm}} \cap L^0_+ = \{0\}$, and 0 is maximal in $\mathcal{G}_{\mathrm{adm}}$ if and only if whenever $g \in \mathcal{G}_{\mathrm{adm}}$ satisfies $g \geqslant 0$ (i.e. $g \in L^0_+$), we have $g = 0$. It is clear that these two statements are equivalent, thus completing the proof.

Exercise 11.2 *(Discrete time: all elements are maximal)* Fix a finite time horizon $T \in \mathbb{N}$, and let $S = (S_k)_{k=0,1,\dots,T}$ be a discrete-time process. Let Θ denote the space of all predictable processes. Show that if S satisfies (NA), then neither $\mathcal{G}_{\mathrm{adm}}$ nor $G_T(\Theta)$ contain any non-maximal element.

Can you also show the result without using Theorem 1.2?

Solution 11.2 As $\mathcal{G}_{\mathrm{adm}} \subseteq G_T(\Theta)$, it suffices to prove that every element of $G_T(\Theta)$ is maximal. So take $G_T(\vartheta) \in G_T(\Theta)$, and suppose $G_T(\vartheta') \in G_T(\Theta)$ such that $G_T(\vartheta) \leqslant G_T(\vartheta')$. Then we have

$$G_T(\vartheta' - \vartheta) = G_T(\vartheta') - G_T(\vartheta) \geqslant 0.$$

Since $\vartheta' - \vartheta \in \Theta$, we have $G_T(\vartheta' - \vartheta) \in G_T(\Theta) \cap L^0_+$. As S satisfies (NA), Theorem 1.2 implies that $G_T(\Theta) \cap L^0_+ = \{0\}$, and hence $G_T(\vartheta' - \vartheta) = 0$. It follows immediately that $G_T(\vartheta) = G_T(\vartheta')$, so that $G_T(\vartheta)$ is maximal in $G_T(\Theta)$, as required.

Another way to get the result is as follows. Since S satisfies (NA), Corollary 1.3 implies that there is an equivalent martingale measure Q for S. Now take $G_T(\vartheta) \in G_T(\Theta)$, and suppose there exists $G_T(\vartheta') \in G_T(\Theta)$ satisfying $G_T(\vartheta') \geqslant G_T(\vartheta)$. Then we have $G_T(\vartheta' - \vartheta) \geqslant 0$, and thus the Q-local martingale $G(\vartheta' - \vartheta)$ is in fact a Q-martingale. This implies that $E[G_T(\vartheta' - \vartheta)] = 0$, and hence $G_T(\vartheta' - \vartheta) = 0$ Q-a.s., and thus also P-a.s. So we have $G_T(\vartheta) = G_T(\vartheta')$, and thus $G_T(\vartheta)$ is maximal in $G_T(\Theta)$. This completes the proof.

Exercise 11.3 *(Uniqueness)* Let S be an RCLL semimartingale satisfying

(NFLVR) and let w be a feasible weight function. Suppose $f \in L^0$ with $|f| \leqslant w$, and assume $\alpha = \beta$ are finite, where as usual $\alpha := \inf \Gamma_+$ and $\beta := \sup \Gamma_-$ are the super- and subreplicating prices for f.

(a) Show that there exists a unique $g \in \mathcal{G}_w$ such that $f \leqslant \alpha + g$.

(b) Show directly (without using any results from the course) that g is maximal in $\mathcal{G}_w$.

Solution 11.3

(a) By Theorem 9.1, there exist $g, g' \in \mathcal{G}_w$ with

$$\alpha + g \geqslant f \quad \text{and} \quad -\beta + g' \geqslant -f.$$

(Note this gives existence of g.) Adding these inequalities and remembering that $\alpha = \beta$, we get $g + g' \geqslant 0$. But also by Lemma 8.1(a), we have $E_Q[g + g'] \leqslant 0$ for any $Q \in \mathbb{P}^w_{e,\sigma}$, so that $g + g' = 0$. We then have

$$\alpha + g \geqslant f \quad \text{and} \quad -\alpha - g \geqslant -f.$$

The second inequality above is equivalent to $f \geqslant \alpha + g$, and together with the first inequality above we get $\alpha + g = f$, so that $g = f - \alpha$. It follows immediately that g is unique, as required.

(b) Assume $g' \in \mathcal{G}$ with $g' \geqslant g$. Then we have $f \leqslant \alpha + g \leqslant \alpha + g'$, and by the uniqueness in part (a), we get $g' = g$ so that g is maximal, as required.

Exercise 11.4 *(Maximality in a larger set)* Let S be an RCLL semimartingale satisfying (NFLVR). Let w be a feasible weight function and fix $g \in \mathcal{G}_w$. Define the random variable $w' := w + g^+$, where $g^+ := \max\{g, 0\}$. Show that w' is a feasible weight function, and that if g is maximal in $\mathcal{G}_w$ then g is also maximal in $\mathcal{G}_{w'}$.

Solution 11.4 Since S satisfies (NFLVR), we know that $\mathbb{P}^w_{\mathrm{e},\sigma} \neq \varnothing$. So take $Q \in \mathbb{P}^w_{\mathrm{e},\sigma}$. By Lemma 8.1(a), we have $E_Q[g] \leqslant 0$, and hence $g^+ \in L^1(Q)$. Since $w' \geqslant w$ and $Q \in \mathbb{P}^w_{\mathrm{e},\sigma} \subseteq \mathbb{P}_{\mathrm{e},\sigma}$, it follows that w' is a feasible weight function.

Now assume g is maximal in $\mathcal{G}_w$. Before showing g is maximal in $\mathcal{G}_{w'}$, we first check that $\mathbb{P}^{w'}_{\mathrm{e},\sigma} = \mathbb{P}^w_{\mathrm{e},\sigma}$. Since $w' \geqslant w$, clearly $\mathbb{P}^{w'}_{\mathrm{e},\sigma} \subseteq \mathbb{P}^w_{\mathrm{e},\sigma}$, and the reverse inclusion follows as $g^+ \in L^1(Q)$ and $Q \in \mathbb{P}^w_{\mathrm{e},\sigma}$ was chosen arbitrarily.

Now using $w' \geqslant w$ and $\mathbb{P}^{w'}_{e,\sigma} = \mathbb{P}^w_{e,\sigma}$, we have $\mathcal{G}_{w'} \supseteq \mathcal{G}_w$, so that in particular $g \in \mathcal{G}_{w'}$. It remains to prove that g is also maximal in $\mathcal{G}_{w'}$. To this end, suppose $\widetilde{g} \in \mathcal{G}_{w'}$ with $\widetilde{g} \geqslant g$. As $g \in \mathcal{G}_w$, there exists some $a \geqslant 0$ such that $g \geqslant -aw$, and thus also $\widetilde{g} \geqslant -aw$. Hence Lemma 8.1(d) gives $\widetilde{g} \in \mathcal{G}^1_{aw} \subseteq \mathcal{G}_w$. As $\widetilde{g} \in \mathcal{G}_w$ and g is maximal in $\mathcal{G}_w$, it follows that $\widetilde{g} = g$, as claimed.

Exercise 11.5 *(An example where $\Gamma_+ \cap \Gamma_-$ is large)* Construct a model for a financial market and a payoff f such that $\Gamma_+(f)$ and $\Gamma_-(f)$ intersect in more than one point.

Solution 11.5 Consider a process $S = (S_0, S_1)$ in discrete time such that $S_0 = 1$ and S_1 satisfies $P[S_1 = \frac{3}{2}] = P[S_1 = 2] = \frac{1}{2}$. Note that for each $b \in \mathbb{R}$ there is an element $G_T(\vartheta_b) \in G_T(\Theta)$ satisfying $P[G_T(\vartheta_b) = b] = P[G_T(\vartheta_b) = \frac{b}{2}] = \frac{1}{2}$, and moreover every element $G_T(\vartheta) \in G_T(\Theta)$ is of this form.

Now define the payoff $f \equiv 0$. A real number $a \in \mathbb{R}$ is an element of $\Gamma_+(f)$ if $a + b \geqslant 0$ and $a + \frac{b}{2} \geqslant 0$ for some $b \in \mathbb{R}$. Thus $\Gamma_+ \supseteq [-\frac{b}{2}, \infty)$ for all $b \geqslant 0$, and hence $\Gamma_+(f) = \mathbb{R}$.

Similarly, a real number $a \in \mathbb{R}$ is an element of $\Gamma_-(f)$ if $a \leqslant b$ and $a \leqslant \frac{b}{2}$ for some $b \in \mathbb{R}$. Thus $\Gamma_- \supseteq (-\infty, \frac{b}{2}]$ for all $b \geqslant 0$, and hence $\Gamma_-(f) = \mathbb{R}$, so that $\Gamma_+(f) \cap \Gamma_-(f) = \mathbb{R}$.