Mathematical Finance

Exercise Sheet 12

Submit by 12:00 on Wednesday, December 20 via the course homepage.

Exercise 12.1 *(Some properties of u)* Let $U : (0, \infty) \to \mathbb{R}$ be a concave and increasing function. Define the function $u : (0, \infty) \to (-\infty, +\infty]$ by

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)],$$

where $\mathcal{V}(x) := \{x + G(\vartheta) : \vartheta \in \Theta^x_{\mathrm{adm}}\}$.

(a) Show that u is concave and increasing.

(b) If additionally $u(x_0) < \infty$ for some $x_0 > 0$, show that $u(x) < \infty$ for all $x > 0$.

Exercise 12.2 *(Utility in a complete market)* Consider a financial market modelled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0, \infty) \to \mathbb{R}$ be a utility function such that $u(x_0) < \infty$ for some $x_0 \in (0, \infty)$.

(a) Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Assume furthermore that $\mathcal{F}_0$ is trivial and fix $z > 0$. Show that $h \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s. for all $h \in \mathcal{D}(z)$, and deduce that

$$j(z) = E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

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(b) Consider the Black–Scholes market $(\widetilde{S}^0, \widetilde{S}^1)$ given by

$$\begin{aligned} \mathrm{d}\tilde{S}^0_0 &= r\tilde{S}^0_t\,\mathrm{d}t, & \tilde{S}^0_0 &= 1,\\ \mathrm{d}\tilde{S}^1_t &= \tilde{S}^1_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), & \tilde{S}^1_0 &= s > 0. \end{aligned}$$

Let $U : (0, \infty) \to \mathbb{R}$ be defined by $U(x) := \frac{1}{\gamma}x^\gamma$, where $\gamma \in (-\infty, 1)\backslash\{0\}$. Show that for $z > 0$,

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu - r)^2 T}{\sigma^2}\right).$$

Exercise 12.3 *(Utility in a market with arbitrage)* Consider a general market with finite time horizon T. Let $U : (0, \infty) \to \mathbb{R}$ be an increasing and concave utility function. Suppose that U is unbounded from above and that either the market admits a 0-admissible arbitrage opportunity, or we are in finite discrete time and the market admits an (admissible) arbitrage opportunity. Show that in both cases, we have $u \equiv \infty$.

Without imposing that U is unbounded from above, what can you say about the relationship between $u(x)$ and $U(x)$ as $x \to \infty$?