Mathematical Finance

Exercise Sheet 12

Submit by 12:00 on Wednesday, December 20 via the course homepage.

Exercise 12.1 *(Some properties of u)* Let $U : (0, \infty) \to \mathbb{R}$ be a concave and increasing function. Define the function $u : (0, \infty) \to (-\infty, +\infty]$ by

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)],$$

where $\mathcal{V}(x) := \{x + G(\vartheta) : \vartheta \in \Theta^x_{\text{adm}}\}$.

(a) Show that u is concave and increasing.

(b) If additionally $u(x_0) < \infty$ for some $x_0 > 0$, show that $u(x) < \infty$ for all $x > 0$.

Solution 12.1

(a) We first prove that u is concave. So fix $x, y \in (0, \infty)$ and $\lambda \in (0, 1)$. We need to show that

$$u\Big(\lambda x + (1 - \lambda) y\Big) \geqslant \lambda u(x) + (1 - \lambda) u(y).$$

First note that if either $u(x)$ or $u(y)$ is $-\infty$, then the inequality holds trivially. So assume that $u(x), u(y) > -\infty$. Take $x + G(\vartheta^x) \in \mathcal{V}(x)$ and $y + G(\vartheta^y) \in \mathcal{V}(y)$ with $U(x + G(\vartheta^x))^-, U(y + G(\vartheta^y))^-$ both in L^1. Then

$$\lambda\Big(x + G(\vartheta^x)\Big) + (1 - \lambda)\Big(y + G(\vartheta^y)\Big) = \lambda x + (1 - \lambda) y + G\Big(\lambda \vartheta^x + (1 - \lambda)\vartheta^y\Big).$$

As U is concave, we have

$$\lambda U\Big(x + G(\vartheta^x)\Big) + (1 - \lambda) U\Big(y + G(\vartheta^y)\Big) \leqslant U\Big(\lambda x + (1 - \lambda) y + G\Big(\lambda \vartheta^x + (1 - \lambda)\vartheta^y\Big)\Big).$$

So also $U(\lambda x + (1 - \lambda) y + G\lambda\vartheta^x + (1 - \lambda)\vartheta^y))^- \in L^1$. Now because we have $\lambda\vartheta^x + (1 - \lambda)\vartheta^y \in \Theta^{\lambda x + (1 - \lambda) y}_{\text{adm}}$, we can take expectations in the above, and this gives

$$\lambda E\Big[U\Big(x + G(\vartheta^x)\Big)\Big] + (1 - \lambda) E\Big[U\Big(y + G(\vartheta^y)\Big)\Big] \leqslant u\Big(\lambda x + (1 - \lambda) y\Big).$$

Finally, taking the supremum over all $x + G(\vartheta^x) \in \mathcal{V}(x)$ and $y + G(\vartheta^y) \in \mathcal{V}(y)$ with integrable negative parts gives the required inequality.

It remains to prove that u is increasing. This follows from the fact that $\Theta^x_{\mathrm{adm}} \subseteq \Theta^y_{\mathrm{adm}}$ for $0 < x < y$. Indeed, for $x + G(\vartheta^x) \in \mathcal{V}(x)$ so that $\vartheta^x \in \Theta^x_{\mathrm{adm}}$, we have $y + G(\vartheta^x) \in \mathcal{V}(y)$, and as U is increasing, this implies

$$E\Big[U\Big(x + G(\vartheta^x)\Big)\Big] \leqslant E\Big[U\Big(y + G(\vartheta^x)\Big)\Big] \leqslant u(y).$$

Taking the supremum over all $\vartheta^x \in \Theta^x_{\mathrm{adm}}$ gives $u(x) \leqslant u(y)$, completing the proof.

(b) As u is increasing, we know that $u(x) < \infty$ for all $x < x_0$. It thus remains to show that $u(x) < \infty$ for all $x > x_0$. By choosing $\lambda \in (0,1)$ small enough, we can find $y \in (0, x_0)$ such that

$$x_0 = \lambda x + (1-\lambda) y.$$

By concavity of u, we have

$$\lambda u(x) + (1-\lambda) u(y) \leqslant u(x_0) < \infty,$$

which gives the result because $u(y) \leqslant u(x_0) < \infty$ and $u(y) \geqslant U(y) > -\infty$.

Exercise 12.2 *(Utility in a complete market)* Consider a financial market modelled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0,\infty) \to \mathbb{R}$ be a utility function such that $u(x_0) < \infty$ for some $x_0 \in (0,\infty)$.

(a) Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Assume furthermore that $\mathcal{F}_0$ is trivial and fix $z > 0$. Show that $h \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s. for all $h \in \mathcal{D}(z)$, and deduce that

$$j(z) = E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

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(b) Consider the Black–Scholes market $(\widetilde{S}^0, \widetilde{S}^1)$ given by

$$\begin{aligned} \mathrm{d}\tilde{S}^0_0 &= r\tilde{S}^0_t\,\mathrm{d}t, & \tilde{S}^0_0 &= 1, \\ \mathrm{d}\tilde{S}^1_t &= \tilde{S}^1_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), & \tilde{S}^1_0 &= s > 0. \end{aligned}$$

Let $U : (0,\infty) \to \mathbb{R}$ be defined by $U(x) := \frac{1}{\gamma}x^\gamma$, where $\gamma \in (-\infty, 1)\backslash\{0\}$. Show that for $z > 0$,

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2 T}{\sigma^2}\right).$$

Solution 12.2

(a) Recall that in general, a payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable if and only if the supremum
$$\sup_{Q^0 \in \mathbb{P}_{e,\sigma}} E_{Q^0}[H]$$
is finite and attained at some $Q^* \in \mathbb{P}_{e,\sigma}$. In our setting, $\mathbb{P}_{e,\sigma}$ is the singleton set $\{Q\}$, so that a payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable if and only if $E_Q[H] < \infty$, i.e. if and only if $H \in L^1_+(Q, \mathcal{F}_T)$.

Now we recall that
$$\mathcal{D}(z) := \{h \in L^0_+(\mathcal{F}_T) : \exists Z \in \mathcal{Z}(z) \text{ with } h \leqslant Z_T\}.$$

So take $h \in \mathcal{D}(z)$ and suppose for contradiction that we do not have $h \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s. Then setting $A := \{h > z\frac{\mathrm{d}Q}{\mathrm{d}P}\}$, we have $P[A] > 0$. Now define the process $M = (M_t)_{0 \leqslant t \leqslant T}$ by
$$M_t := E_Q[\mathbf{1}_A \mid \mathcal{F}_t].$$

Then M is a nonnegative Q-martingale with $M_0 = Q[A] > 0$ because $Q \approx P$. Since $E_Q[M_T] \leqslant 1 < \infty$, it follows that $M_T \in L^0_+(\mathcal{F}_T)$ is attainable so that there exists some $\vartheta \in \Theta_{\mathrm{adm}}$ with
$$M = M_0 + G(\vartheta).$$

Since M is nonnegative, we must have $\vartheta \in \Theta^{M_0}_{\mathrm{adm}}$ and hence $M \in \mathcal{V}(M_0)$.

Now, since $h \in \mathcal{D}(z)$, there exists $Z \in \mathcal{Z}(z)$ such that $h \leqslant Z_T$. By the definition of $\mathcal{Z}(z)$, the product ZM is a P-supermartingale. We thus have
$$E[hM_T] \leqslant E[Z_T M_T] \leqslant E[Z_0 M_0] = zM_0.$$

Also, we have $E[z\frac{\mathrm{d}Q}{\mathrm{d}P} M_T] = E_Q[zM_T] = zM_0$, and thus
$$E\left[\left(h - z\frac{\mathrm{d}Q}{\mathrm{d}P}\right) M_T\right] \leqslant 0.$$

But recalling $M_T = \mathbf{1}_A$ and $P[A] > 0$ gives
$$E\left[\left(h - z\frac{\mathrm{d}Q}{\mathrm{d}P}\right) M_T\right] > 0,$$

which gives a contradiction. Hence we must have $h \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s., as required. In particular, as any $Z_T \in \mathcal{D}(z)$ for $Z \in \mathcal{Z}(z)$, this gives $Z_T \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ for any $Z \in \mathcal{Z}(z)$.

It remains to show $j(z) = E[J(z\frac{\mathrm{d}Q}{\mathrm{d}P})]$. First we recall
$$j(z) := \inf_{Z \in \mathcal{Z}(z)} E[J(Z_T)].$$

For each $Z \in \mathcal{Z}(z)$ we have $Z_T \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$. As J is decreasing, we have

$$J(Z_T) \geqslant J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right),$$

and thus

$$E[J(Z_T)] \geqslant E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

Taking the infimum over all $Z \in \mathcal{Z}(z)$ gives

$$j(z) \geqslant E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

As $z\frac{\mathrm{d}Q}{\mathrm{d}P} \in \mathcal{Z}(z)$, this concludes the proof.

(b) In the Black–Scholes model, there exists a unique EMM Q, and thus part (a) is applicable. We hence have

$$j(z) = E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

To compute this, we start by writing

$$J(y) = \sup_{x>0}\Big(U(x) - xy\Big) = \sup_{x>0}\left(\frac{1}{\gamma}x^\gamma - xy\right).$$

Taking the derivative of $\frac{1}{\gamma}x^\gamma - xy$ with respect to x and setting it equal to zero, we get $x = y^{\frac{1}{\gamma-1}}$, and hence

$$J(y) = \frac{1}{\gamma}y^{\frac{\gamma}{\gamma-1}} - y^{\frac{\gamma}{\gamma-1}} = \frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}}.$$

We also recall that in the Black–Scholes model,

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \mathcal{E}(-\lambda W)_T,$$

where $\lambda := \frac{\mu - r}{\sigma}$. So we have

$$\begin{aligned}
j(z) &= E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right] \\
&= \frac{1-\gamma}{\gamma}E\left[\mathcal{E}(-\lambda W)_T^{\frac{\gamma}{\gamma-1}}\right] \\
&= \frac{1-\gamma}{\gamma}z^{\frac{\gamma}{\gamma-1}}E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}W_T + \frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right)\right] \\
&= \frac{1-\gamma}{\gamma}z^{\frac{\gamma}{\gamma-1}}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right)E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma}W\right)_T\right] \\
&= \frac{1-\gamma}{\gamma}z^{\frac{\gamma}{\gamma-1}}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right),
\end{aligned}$$

where in the last step we use that $\mathcal{E}(aW)$ is a P-martingale for each $a \in \mathbb{R}$. Substituting $\lambda = \frac{\mu - r}{\sigma}$ then gives the result.

Exercise 12.3 *(Utility in a market with arbitrage)* Consider a general market with finite time horizon T. Let $U : (0, \infty) \to \mathbb{R}$ be an increasing and concave utility function. Suppose that U is unbounded from above and that either the market admits a 0-admissible arbitrage opportunity, or we are in finite discrete time and the market admits an (admissible) arbitrage opportunity. Show that in both cases, we have $u \equiv \infty$.

Without imposing that U is unbounded from above, what can you say about the relationship between $u(x)$ and $U(x)$ as $x \to \infty$?

Solution 12.3 By assumption, there exists $\vartheta \in \Theta_{\mathrm{adm}}$ such that $G_T(\vartheta) \geqslant 0$ P-a.s. and $P[G_T(\vartheta) > 0] > 0$. By Exercise 4.2, we may assume that ϑ is 0-admissible, and so also $n\vartheta$ is 0-admissible for each $n \in \mathbb{N}$. It follows that $x + nG_T(\vartheta) \in \mathcal{V}(x)$ for every $x > 0$ and $n \in \mathbb{N}$. So setting $A := \{G_T(\vartheta) > 0\}$, we have that for all $x > 0$ and $n \in \mathbb{N}$,

$$u(x) \geqslant E\Big[U\Big(x + nG_T(\vartheta)\Big)\Big] = E\Big[U\Big(x + nG_T(\vartheta)\Big)\mathbf{1}_A\Big] + E\Big[U(x)\mathbf{1}_{A^c}\Big].$$

As U is increasing, we can let $n \to \infty$ and apply the monotone convergence theorem to get that for all $x > 0$,

$$u(x) \geqslant E\Big[U(\infty)\mathbf{1}_A\Big] + E\Big[U(x)\mathbf{1}_{A^c}\Big].$$

Note that U is increasing gives that the limit $U(\infty) := \lim_{x\to\infty} U(x) \in \mathbb{R} \cup \{\infty\}$ exists. Since U is unbounded from above we have $U(\infty) = \infty$, and as $P[A] > 0$, we can conclude that $u \equiv \infty$, as required.

Now suppose that U is not necessarily unbounded from above. We still have

$$u(x) \geqslant E\Big[U(\infty)\mathbf{1}_A\Big] + E\Big[U(x)\mathbf{1}_{A^c}\Big] = U(\infty)P[A] + U(x)P[A^c].$$

Also, by the definition of u, $u(x) \leqslant U(\infty)$ as U is increasing. So for each $x > 0$,

$$U(\infty)P[A] + U(x)P[A^c] \leqslant u(x) \leqslant U(\infty).$$

Letting $x \to \infty$ in the above gives $u(\infty) = U(\infty)$. This completes the problem.