Mathematical Finance

Exercise Sheet 13

Submit by 12:00 on Wednesday, December 27 via the course homepage.

Exercise 13.1 *(Utility in a complete market)* Consider a financial market modelled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0, \infty) \to \mathbb{R}$ be a utility function such that $u(x) < \infty$ for some (and hence for all) $x \in (0, \infty)$. Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Assume furthermore that $\mathcal{F}_0$ is trivial.

(a) Let $z_0 := \inf\{z > 0 : j(z) < \infty\}$. Show that the function j defined in the lecture notes is in $C^1((z_0, \infty); \mathbb{R})$ and satisfies

$$j'(z) = E\left[\frac{\mathrm{d}Q}{\mathrm{d}P} J'\left(z \frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right], \quad z \in (z_0, \infty).$$

(b) Set $x_0 := \lim_{z \downarrow z_0}(-j'(z))$ and fix $x \in (0, x_0)$. Let $z_x \in (z_0, \infty)$ be the unique number such that $-j'(z_x) = x$. Show that $f^* := I(z_x \frac{\mathrm{d}Q}{\mathrm{d}P})$ is the unique solution to the primal problem

$$u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)].$$

Exercise 13.2 *(The Merton problem)* Consider the Black–Scholes market given by

$$\begin{aligned} \mathrm{d}\tilde{S}^0_0 &= r \tilde{S}^0_t \, \mathrm{d}t, & \tilde{S}^0_0 &= 1, \\ \mathrm{d}\tilde{S}^1_t &= \tilde{S}^1_t (\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t), & \tilde{S}^1_0 &= s > 0. \end{aligned}$$

Let $U : (0, \infty) \to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma} x^\gamma$, where $\gamma \in (-\infty, 1) \backslash \{0\}$. Recall from Exercise 12.2(b) that

$$j(z) = \frac{1 - \gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2} \frac{\gamma}{(1-\gamma)^2} \frac{(\mu - r)^2}{\sigma^2} T\right), \quad z \in (0, \infty).$$

We consider the *Merton problem* of maximising expected utility from final wealth (in units of $\widetilde{S}^0$).

(a) Show that the unique solution to the primal problem

$$u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)], \quad x \in (0, \infty),$$

is given by $f_x^* := x\mathcal{E}(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R)_T$, where the process $R = (R_t)_{0\leqslant t\leqslant T}$ is defined by $R_t = W_t + \frac{\mu-r}{\sigma}t$.

(b) Deduce that $f_x^* = V_T(x, \vartheta^x)$, where the integrand $\vartheta^x = (\vartheta_t^x)_{0\leqslant t\leqslant T}$ is given by

$$\vartheta_t^x = \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t, \quad x \in (0,\infty),$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad x \in (0,\infty).$$

(c) For any x-admissible ϑ with $V(x,\vartheta) > 0$, denote by

$$\pi_t := \frac{\vartheta_t S_t^1}{V_t(x,\vartheta)}$$

the fraction of wealth that is invested in the stock. Show that the optimal strategy ϑ^x is given by the *Merton proportion*

$$\pi_t^* = \frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}.$$

Exercise 13.3 *($\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ has moments of all orders)* Let S be a continuous real-valued semimartingale satisfying the structure condition (SC), i.e. there exist a continuous local martingale M null at zero and a predictable process λ such that

$$S = S_0 + M + \int \lambda \, \mathrm{d}\langle M\rangle,$$

and with the mean-variance tradeoff process $K = \int \lambda^2 \, \mathrm{d}\langle M\rangle$ bounded. Now define $\hat{Z} := \mathcal{E}(-\lambda \bullet M)$ and $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \hat{Z}_T$.

(a) Show that $\hat{P} \in \mathbb{P}_{\mathrm{e,loc}}(S)$.

(b) Show that both $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $\frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$ have moments of all orders.