Mathematical Finance

Exercise Sheet 13

Submit by 12:00 on Wednesday, December 27 via the course homepage.

Exercise 13.1 *(Utility in a complete market)* Consider a financial market modelled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0,\infty) \to \mathbb{R}$ be a utility function such that $u(x) < \infty$ for some (and hence for all) $x \in (0,\infty)$. Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Assume furthermore that $\mathcal{F}_0$ is trivial.

(a) Let $z_0 := \inf\{z > 0 : j(z) < \infty\}$. Show that the function j defined in the lecture notes is in $C^1((z_0,\infty);\mathbb{R})$ and satisfies

$$j'(z) = E\left[\frac{\mathrm{d}Q}{\mathrm{d}P} J'\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right], \quad z \in (z_0,\infty).$$

(b) Set $x_0 := \lim_{z\downarrow z_0}(-j'(z))$ and fix $x \in (0, x_0)$. Let $z_x \in (z_0,\infty)$ be the unique number such that $-j'(z_x) = x$. Show that $f^* := I(z_x \frac{\mathrm{d}Q}{\mathrm{d}P})$ is the unique solution to the primal problem

$$u(x) = \sup_{f\in\mathcal{C}(x)} E[U(f)].$$

Solution 13.1 For notational convenience, we denote by $Z = (Z_t)_{0\leqslant t\leqslant T}$ the density process of Q with respect to P, so that $Z_T = \frac{\mathrm{d}Q}{\mathrm{d}P}$.

(a) Note that $0 \leqslant z_0 < \infty$ by Theorem 12.4, and also by Theorem 12.4, we have that $j(z) < \infty$ for $z \in (z_0,\infty)$.

Now recall that J is in C^1 and strictly decreasing. We can thus define the function $g : (z_0,\infty) \to [-\infty, 0]$ by

$$g(s) = E[Z_T J'(sZ_T)].$$

Moreover, as J is also strictly convex, J' is increasing, and thus g is also increasing since $Z_T > 0$. As g is negative-valued, it follows from the dominated convergence theorem that if $g(s_0) > -\infty$ for some $s_0 > z_0$, we have that g is continuous on (s_0,∞).

Next, since $\frac{\mathrm{d}}{\mathrm{d}s}J(sZ_T) = Z_T J'(sZ_T)$ by the chain rule, we have by the fundamental theorem of calculus that for $z_0 < z_1 < z_2 < \infty$,

$$J(z_2 Z_T) - J(z_1 Z_T) = \int_{z_1}^{z_2} Z_T J'(sZ_T)\,\mathrm{d}s.$$

By Exercise 12.2(a), we know that $j(z) = E[J(zZ_T)]$. Thus taking expectations of both sides in the above gives

$$j(z_2) - j(z_1) = E\left[\int_{z_1}^{z_2} Z_T J'(sZ_T)\,\mathrm{d}s\right] = \int_{z_1}^{z_2} E[Z_T J'(sZ_T)]\,\mathrm{d}s = \int_{z_1}^{z_2} g(s)\,\mathrm{d}s,$$

where the second step uses the Fubini–Tonelli theorem, keeping in mind that the integrand is strictly negative.

Note that by the definition of z_0, we have that $j(z_2) - j(z_1)$ is finite, and thus the function g is finite a.e. on (z_0, ∞). From the above, we can conclude that g is continuous and finite on (z_0, ∞). By dividing by $z_2 - z_1$ and letting $z_2 \to z_1$, we get that

$$j'(z) = E[Z_T J'(zZ_T)] = g(z)$$

as required. Now since g is continuous on (z_0, ∞), we have $j \in C^1((z_0, \infty); \mathbb{R})$, completing the proof.

(b) Before establishing that f^* is a solution to the primal problem, we first need to check that $f^* \in \mathcal{C}(x)$. To this end, recall that $f \in \mathcal{C}(x)$ if and only if

$$\sup_{h \in \mathcal{D}(1)} E[fh] \leqslant x.$$

By Exercise 12.2(a), this is equivalent to

$$E[fZ_T] \leqslant x.$$

Now by the definition of f^* and I, we have

$$E[f^* Z_T] = E[I(z_x Z_T) Z_T] = E[-J'(z_x Z_T) Z_T].$$

Moreover, by part (a), we have $E[Z_T J'(z_x Z_T)] = j'(z_x)$, and since $-j'(z_x) = x$ by definition of z_x, we have

$$E[f^* Z_T] = x$$

and thus in particular $f^* \in \mathcal{C}(x)$, as required.

Next, we establish that f^* is a solution to the primal problem. So fix $f \in \mathcal{C}(x)$. We need to show that $E[U(f^*)] \geqslant E[U(f)]$. We may thus assume without loss of generality that $E[U(f)] > -\infty$. Now since U is in C^1 and strictly concave on $(0, \infty)$, and since $f^* > 0$ P-a.s., we have

$$U(f) - U(f^*) \leqslant U'(f^*)(f - f^*),$$

with strict inequality on the event $\{f \neq f^*\}$. Now note that

$$U'(f^*) = U'\Big(I(z_x Z_T)\Big) = z_x Z_T.$$

Thus taking expectations of the above inequality yields

$$E\Big[U(f) - U(f^*)\Big] \leqslant E\Big[z_x Z_T (f - f^*)\Big],$$

and since $E[Z_T f^*] = x$ and $E[Z_T f] \leqslant x$ and $z_x > 0$, we have

$$E[U(f) - U(f^*)] \leqslant 0,$$

and the inequality is strict when $P[f \neq f^*] > 0$. It follows immediately that f^* is the unique solution to the primal problem. This completes the proof.

Exercise 13.2 *(The Merton problem)* Consider the Black–Scholes market given by

$$\begin{aligned} \mathrm{d}\tilde{S}_0^0 &= r\tilde{S}_t^0 \,\mathrm{d}t, & \tilde{S}_0^0 &= 1, \\ \mathrm{d}\tilde{S}_t^1 &= \tilde{S}_t^1(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t), & \tilde{S}_0^1 &= s > 0. \end{aligned}$$

Let $U : (0,\infty) \to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma}x^\gamma$, where $\gamma \in (-\infty, 1)\backslash\{0\}$. Recall from Exercise 12.2(b) that

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad z \in (0,\infty).$$

We consider the *Merton problem* of maximising expected utility from final wealth (in units of $\widetilde{S}^0$).

(a) Show that the unique solution to the primal problem

$$u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)], \quad x \in (0,\infty),$$

is given by $f_x^* := x\mathcal{E}(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R)_T$, where the process $R = (R_t)_{0\leqslant t\leqslant T}$ is defined by $R_t = W_t + \frac{\mu-r}{\sigma}t$.

(b) Deduce that $f_x^* = V_T(x, \vartheta^x)$, where the integrand $\vartheta^x = (\vartheta_t^x)_{0\leqslant t\leqslant T}$ is given by

$$\vartheta_t^x = \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t, \quad x \in (0,\infty),$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad x \in (0,\infty).$$

(c) For any x-admissible ϑ with $V(x,\vartheta) > 0$, denote by

$$\pi_t := \frac{\vartheta_t S_t^1}{V_t(x,\vartheta)}$$

the fraction of wealth that is invested in the stock. Show that the optimal strategy ϑ^x is given by the *Merton proportion*

$$\pi_t^* = \frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}.$$

Solution 13.2

(a) First, note that $j(z) < \infty$ for some $z \in (0,\infty)$ implies that

$$u(x) \leqslant j(z) + zx < \infty, \quad x \in (0,\infty).$$

In Exercise 12.2(b) we computed $J(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}}$, and hence $J'(z) = -z^{-\frac{1}{1-\gamma}}$. Now fix $x > 0$. With the same notation as in Exercise 13.1, we have

$$\begin{aligned} f_x^* &= -J'\left(z_x \frac{dQ}{dP}\right) = z_x^{-\frac{1}{1-\gamma}} \left(\mathcal{E}(-\lambda W)_T\right)^{-\frac{1}{1-\gamma}} \\ &= -j'(z_x) \exp\left(-\frac{1}{2}\frac{\lambda^2 \gamma}{(1-\gamma)^2} T\right) \exp\left(\frac{\lambda}{1-\gamma} W_T + \frac{1}{2}\frac{\lambda^2}{1-\gamma} T\right) \\ &= x \exp\left(\frac{\lambda}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2}{(1-\gamma)^2} T\right) \\ &= x\mathcal{E}\left(\frac{\lambda}{1-\gamma} R\right)_T. \end{aligned}$$

This completes the proof.

(b) Fix $x > 0$. By the definition of the stochastic exponential and using that $\lambda = \frac{\mu - r}{\sigma}$, we have

$$\begin{aligned} f_x^* &= x\left(1 + \int_0^T \mathcal{E}\left(\frac{\lambda}{1-\gamma} R\right)_t \frac{\lambda}{1-\gamma}\, \mathrm{d}R_t\right) \\ &= x + \int_0^T x\mathcal{E}\left(\frac{\lambda}{1-\gamma} R\right)_t \frac{\lambda}{1-\gamma}\frac{1}{\sigma S_t^1}\, \mathrm{d}S_t^1 \\ &= x + \int_0^T x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma} R\right)_t \frac{1}{1-\gamma}\frac{\mu - r}{\sigma}\frac{1}{\sigma S_t^1}\, \mathrm{d}S_t^1 \\ &= x + \int_0^T \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma} R\right)_t \mathrm{d}S_t^1. \end{aligned}$$

This gives the first claim. Now using again that $\mathcal{E}(aW)$ is a P-martingale for all $a \in \mathbb{R}$ and that $\lambda = \frac{\mu - r}{\sigma}$, we have

$$\begin{aligned} u(x) &= E\left[U(f_x^*)\right] = \frac{x^\gamma}{\gamma} E\left[\left(\mathcal{E}\Big(\frac{\lambda}{1-\gamma} R\Big)_T\right)^\gamma\right] \\ &= \frac{x^\gamma}{\gamma} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2 \gamma}{(1-\gamma)^2} T\right)\right] \\ &= \frac{x^\gamma}{\gamma} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma} T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma} W\right)_T\right] \\ &= \frac{x^\gamma}{\gamma} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma} T\right) \\ &= \frac{x^\gamma}{\gamma} \exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu - r)^2}{\sigma^2} T\right). \end{aligned}$$

This completes the proof.

(c) By part (a) and since $\lambda = \frac{\mu - r}{\sigma}$, we have

$$V_t(x, \vartheta^x) = x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma}R\right)_t,$$

and by part (b), we have

$$\vartheta_t^x = \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma}R\right)_t.$$

Therefore, we obtain directly that

$$\pi_t^* := \frac{\vartheta_t^x S_t^1}{V_t(x, \vartheta^x)} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}.$$

This completes the proof.

Exercise 13.3 *($\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ has moments of all orders)* Let S be a continuous real-valued semimartingale satisfying the structure condition (SC), i.e. there exist a continuous local martingale M null at zero and a predictable process λ such that

$$S = S_0 + M + \int \lambda \,\mathrm{d}\langle M \rangle,$$

and with the mean-variance tradeoff process $K = \int \lambda^2 \,\mathrm{d}\langle M \rangle$ bounded. Now define $\hat{Z} := \mathcal{E}(-\lambda \bullet M)$ and $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \hat{Z}_T$.

(a) Show that $\hat{P} \in \mathbb{P}_{\mathrm{e,loc}}(S)$.

(b) Show that both $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $\frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$ have moments of all orders.

Solution 13.3

(a) We need to show that $\hat{P}$ is an equivalent probability measure, and that S is a $\hat{P}$-local martingale. To this end, first note that since K is bounded, we have that

$$E\left[\exp\left(\frac{1}{2}\langle -\lambda \bullet M \rangle_T\right)\right] = E\left[\exp\left(\frac{1}{2}K_T\right)\right] < \infty.$$

So by Novikov's condition, we can conclude that $\hat{Z}$ is a martingale. As $\hat{Z}$ is strictly positive, it follows that $\hat{P}$ is an equivalent probability measure. It now remains to show that S is a $\hat{P}$-local martingale. To this end, we first apply the stochastic product rule to $\hat{Z}S$ and write

$$\mathrm{d}(\hat{Z}S) = \hat{Z}\,\mathrm{d}S + S\,\mathrm{d}\hat{Z} + \mathrm{d}\langle \hat{Z}, S \rangle.$$

Then we use that S satisfies (SC) and that

$$\mathrm{d}\hat{Z} = \mathrm{d}\mathcal{E}(-\lambda \bullet M) = \mathcal{E}(-\lambda \bullet M)\,\mathrm{d}(-\lambda \bullet M) = -\lambda\mathcal{E}(-\lambda \bullet M)\,\mathrm{d}M = -\lambda\hat{Z}\,\mathrm{d}M$$

to compute

$$\begin{aligned}\mathrm{d}(\hat{Z}S) &= \hat{Z}\,\mathrm{d}M + \hat{Z}\lambda\,\mathrm{d}\langle M\rangle - S\lambda\hat{Z}\,\mathrm{d}M - \lambda\hat{Z}\,\mathrm{d}\,\langle M\rangle \\ &= (\hat{Z} - S\lambda\hat{Z})\,\mathrm{d}M.\end{aligned}$$

As $\hat{Z}$, S and M are continuous, it follows that $\hat{Z}S$ is a P-local martingale, so that S is a $\hat{P}$-local martingale, and hence $\hat{P} \in \mathbb{P}_{\mathrm{e,loc}}$, as required.

(b) We compute, for any $p \in \mathbb{R}$,

$$\begin{aligned}\hat{Z}_T^p &= \exp\Big(-p\lambda \bullet M_T - \frac{1}{2}p\lambda^2 \bullet \langle M\rangle_T\Big) \\ &= \exp\Big(-p\lambda \bullet M_T - \frac{1}{2}p^2\lambda^2 \bullet \langle M\rangle_T\Big)\exp\Big(\frac{1}{2}(p^2-p)\lambda^2 \bullet \langle M\rangle_T\Big) \\ &= \mathcal{E}(-p\lambda \bullet M)_T \exp\big((p^2-p)K_T\big).\end{aligned}$$

So letting $C < \infty$ be a bound on K, we can write

$$E[\hat{Z}_T^p] \leqslant E[\mathcal{E}(-p\lambda \bullet M)_T]\exp(C|p^2-p|) \leqslant \exp(C|p^2-p|) < \infty,$$

since $\mathcal{E}(-p\lambda \bullet M)$ is a supermartingale. As $Z_T = \frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $Z_T^{-1} = \frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$, this completes the proof.