Mathematical Finance

Exercise Sheet 1

Submit by 12:00 on Wednesday, September 27 via the course homepage.

Exercise 1.1 *(Path regularity and measurability)* Let $S = (S_t)_{t \geqslant 0}$ be a real-valued stochastic process. Define the processes S^* and A by $S^*_t := \sup_{0 \leqslant r \leqslant t} S_r$ and $A_t := \int_0^t S_r \,\mathrm{d}r$ (when it exists), respectively.

(a) Show that if S is RCLL, then S^* is RCLL and A is well defined and continuous.

Fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions.

(b) Show that if S is RCLL and adapted, then also S^* and A are adapted.

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and define the process $\vartheta = (\vartheta_t)_{t \geqslant 0}$ by $\vartheta_t := f(S_t, S^*_t, A_t)$.

Show that if S is adapted and continuous, then ϑ is predictable.

Solution 1.1

(a) Throughout this part, a single $\omega \in \Omega$ is fixed.

We first show that S^* is RCLL. Fix some $t_0 \in [0, \infty)$. Since S^* is nondecreasing by definition, the left and right limits $\lim_{t \uparrow t_0} S^*_t$ and $\lim_{t \downarrow t_0} S^*_t$ exist, and $\lim_{t \downarrow t_0} S^*_t \geqslant S^*_{t_0}$. It remains to show that $\lim_{t \downarrow t_0} S^*_t \leqslant S^*_{t_0}$. To this end, fix $\varepsilon > 0$ and note that by right-continuity of S, there exists $\delta > 0$ such that $|S_t - S_{t_0}| < \varepsilon$ for all $t \in [t_0, t_0 + \delta]$. It follows that for all $t \in [t_0, t_0 + \delta]$, we have

$$
\begin{aligned}
S^*_t \leqslant S^*_{t_0+\delta} = \sup_{0 \leqslant r \leqslant t_0+\delta} S_r &= \max\left\{ \sup_{0 \leqslant r \leqslant t_0} S_r, \sup_{t_0 \leqslant r \leqslant t_0+\delta} S_r \right\} \\
&\leqslant \max\left\{ \sup_{0 \leqslant r \leqslant t_0} S_r, S_{t_0} + \varepsilon \right\} \leqslant \sup_{0 \leqslant r \leqslant t_0} S_r + \varepsilon = S^*_{t_0} + \varepsilon.
\end{aligned}
$$

Hence $\lim_{t \downarrow t_0} S^*_t \leqslant S^*_{t_0} + \varepsilon$, and letting $\varepsilon \downarrow 0$ gives $\lim_{t \downarrow t_0} S^*_t \leqslant S^*_{t_0}$, as required.

For A, note first that any RCLL function has at most countably many discontinuities on any compact interval, and these form a null set for Lebesgue measure. So it is enough to integrate S only over those $r \in [0, t]$ where it is continuous, and then clearly A_t is well defined.

It remains to show that A is continuous. To this end, fix $0 \leqslant s < t < \infty$, and note that

$$|A_t - A_s| = \left| \int_s^t S_r \,\mathrm{d}r \right| \leqslant \int_s^t |S_r| \,\mathrm{d}r \leqslant |t-s| \sup_{s \leqslant r \leqslant t} |S_r|.$$

Since RCLL functions are locally bounded, $\sup_{s \leqslant r \leqslant t} |S_r|$ exists and is finite. It follows that A is continuous.

(b) We first show that S^* is adapted. Fix $t \geqslant 0$. Since S is right-continuous,

$$S_t^* = \sup_{0 \leqslant r \leqslant t} S_r = \sup_{0 \leqslant r \leqslant t,\, r \in \mathbb{Q} \cup \{t\}} |S_r|.$$

Using that S is adapted, it follows immediately that S_t^* is $\mathcal{F}_t$-measurable, and thus S^* is adapted.

It remains to show that A is adapted. To this end, consider for each $n \in \mathbb{N}$ the process $S^{(n)}$ defined by

$$S_t^{(n)} := \sum_{k=1}^{\infty} \mathbf{1}_{\{\frac{k-1}{n} < t \leqslant \frac{k}{n}\}} S_{\frac{k}{n}}.$$

By (a.s.) right-continuity of S, we have $S_t^{(n)} \to S_t$ as $n \to \infty$ (a.s.). Since RCLL functions are bounded on compact intervals (here $[0, t]$), we can use the dominated convergence theorem to get

$$A_t = \int_0^t S_r \,\mathrm{d}r = \lim_{n \to \infty} \int_0^t S_r^{(n)} \,\mathrm{d}r = \lim_{n \to \infty} \sum_{k=1}^{\lceil nt \rceil} \frac{1}{n} S_{\frac{k}{n}} = \lim_{n \to \infty} \sum_{k=1}^{\lceil nt \rceil - 1} \frac{1}{n} S_{\frac{k}{n}},$$

where the above limit is understood to hold almost surely. Since each sum $\sum_{k=1}^{\lceil nt \rceil - 1} \frac{1}{n} S_{\frac{k}{n}}$ is $\mathcal{F}_t$-measurable, so is the limit $\lim_{n \to \infty} \sum_{k=1}^{\lceil nt \rceil - 1} \frac{1}{n} S_{\frac{k}{n}}$. But A_t is equal to this limit up to a null set; so by completeness of $\mathbb{F}$, A_t is also $\mathcal{F}_t$-measurable, and hence A is adapted, as required.

(c) Since the processes S, S^*, A are adapted, so is ϑ.

Assume first that S is continuous. By repeating the argument in part (a), we can show that S^* is also continuous. We also know from part (a) that A is continuous, and thus so is ϑ. It now follows that ϑ is predictable, since it is an adapted and continuous process.

Exercise 1.2 *(Geometric Brownian motion)* Fix constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion. Define the process $S = (S_t)_{t \geqslant 0}$ by

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

The process $S = (S_t)_{t \geqslant 0}$ is called a *geometric Brownian motion* and is the stock price process in the *Black–Scholes model.*

Find $\lim_{t\to\infty} S_t$ (if it exists) for all possible parameter constellations.

Hint: You may use the law of the iterated logarithm.

Solution 1.2 We can rewrite S_t as

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{2t\log\log t}\frac{W_t}{\sqrt{2t\log\log t}}\right) \\ &= S_0 \exp\left(\sqrt{2t\log\log t}\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{t}{\sqrt{2t\log\log t}} + \sigma\frac{W_t}{\sqrt{2t\log\log t}}\right)\right). \end{aligned}$$

Since

$$\lim_{t\to\infty}\sqrt{2t\log\log t} = \infty \qquad \text{and} \qquad \lim_{t\to\infty}\frac{t}{\sqrt{2t\log\log t}} = \infty$$

and by the law of the iterated logarithm, it follows that

- when $\mu > \frac{\sigma^2}{2}$,
$$\lim_{t\to\infty} S_t = \infty;$$
- when $\mu < \frac{\sigma^2}{2}$,
$$\lim_{t\to\infty} S_t = 0;$$
- when $\mu = \frac{\sigma^2}{2}$,
$$\liminf_{t\to\infty} S_t = 0 \qquad \text{and} \qquad \limsup_{t\to\infty} S_t = \infty,$$
  and hence $\lim_{t\to\infty} S_t$ does not exist.

Alternative solution: We can write

$$\log\frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t.$$

By the law of the iterated logarithm, W_t grows more slowly than t as $t \to \infty$. So for $\mu > \frac{\sigma^2}{2}$, $\log(S_t/S_0) \to \infty$, hence $S_t \to \infty$, and for $\mu < \frac{\sigma^2}{2}$, $\log(S_t/S_0) \to -\infty$, hence $S_t \to 0$. For $\mu = \frac{\sigma^2}{2}$, $\log(S_t/S_0) = \sigma W_t$ has $\limsup_{t\to\infty} \sigma W_t = \infty$ and $\liminf_{t\to\infty} \sigma W_t = -\infty$, hence $\limsup_{t\to\infty} S_t = \infty$ and $\liminf_{t\to\infty} S_t = 0$, so that $\lim_{t\to\infty} S_t$ does not exist.

Exercise 1.3 *(Reparametrisation, Lemma 0.1(2))* Fix a finite time horizon $T > 0$ and let $S = (S_t)_{0\leqslant t\leqslant T}$ be a semimartingale. Prove that there is a bijection between self-financing strategies $\varphi = (\varphi^0, \vartheta)$ and pairs

$$(v_0, \vartheta) \in L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}.$$

Give explicitly the bijection map and its inverse.

Solution 1.3 Recall that a self-financing strategy is a pair $\varphi = (\varphi^0, \vartheta)$, where φ^0 is an adapted real-valued process and ϑ is a predictable, $\mathbb{R}^d$-valued and S-integrable process such that

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_r \, \mathrm{d}S_r, \qquad \forall t \in [0, T].$$

Now consider the map f defined on the family of self-financing strategies that sends each $\varphi = (\varphi^0, \vartheta)$ to (v_0, ϑ), where

$$v_0 = V_0(\varphi) = \varphi_0^0 + \vartheta_0^{\mathrm{tr}} S_0.$$

It is clear that $v_0 \in L^0(\mathcal{F}_0)$, and thus f is a well-defined map on the set of self-financing strategies into the space $L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$. We show that f is a bijection.

Now consider the map g (which we show below to be the inverse of f) defined on the space $L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$ that sends each (v_0, ϑ) to (φ^0, ϑ), where the process φ^0 is defined by

$$\varphi^0 = v_0 + \int \vartheta \, \mathrm{d}S - \vartheta^{\mathrm{tr}} S.$$

It is immediate that φ^0 is a real-valued adapted process. Moreover,

$$\begin{aligned} V_t(\varphi^0, \vartheta) - V_0(\varphi^0, \vartheta) &= \varphi_t^0 + \vartheta_t^{\mathrm{tr}} S_t - \varphi_0^0 - \vartheta_0^{\mathrm{tr}} S_0 \\ &= v_0 + \int_0^t \vartheta_r \, \mathrm{d}S_r - \vartheta_t^{\mathrm{tr}} S_t + \vartheta_t^{\mathrm{tr}} S_t - v_0 + \vartheta_0^{\mathrm{tr}} S_0 - \vartheta_0^{\mathrm{tr}} S_0 \\ &= \int_0^t \vartheta_r \, \mathrm{d}S_r. \end{aligned}$$

It follows that g is a well-defined map into the set of self-financing strategies.

To see that f is a bijection onto the set $L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$, it suffices to show that f and g are inverses of each other. To this end, take a self-financing strategy $\varphi = (\varphi^0, \vartheta)$, and compute

$$\begin{aligned} g \circ f(\varphi) &= g(\phi_0^0 + \vartheta_0^{\mathrm{tr}} S_0, \vartheta) = (\varphi_0^0 + \vartheta_0^{\mathrm{tr}} S_0 + \int \vartheta \, \mathrm{d}S - \vartheta^{\mathrm{tr}} S, \vartheta) \\ &= (\varphi_0^0, \vartheta) = \varphi, \end{aligned}$$

where in the last step we used that φ is self-financing so that

$$\int \vartheta \, \mathrm{d}S = V(\varphi) - V_0(\varphi) = \vartheta^{\mathrm{tr}} S - \vartheta_0^{\mathrm{tr}} S_0.$$

We have thus shown that $g \circ f$ is the identity map.

Similarly, take $(v_0, \vartheta) \in L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$ and compute

$$\begin{aligned} f \circ g(v_0, \vartheta) &= f\left(v_0 + \int \varphi \, \mathrm{d}S - \vartheta^{\mathrm{tr}} S, \vartheta\right) = (v_0 - \vartheta_0^{\mathrm{tr}} S_0 + \vartheta_0^{\mathrm{tr}} S_0, \vartheta) \\ &= (v_0, \vartheta). \end{aligned}$$

Hence, $f \circ g$ is also the identity map, and this completes the proof.