Mathematical Finance

Exercise Sheet 2

Submit by 12:00 on Wednesday, October 4 via the course homepage.

Exercise 2.1 *(From local martingale to supermartingale)* Let $(X_t)_{t \geqslant 0}$ be a local martingale null at zero and $(Y_t)_{t \geqslant 0}$ a martingale such that $Y_t \leqslant X_t$ for each $t \geqslant 0$. Prove that X is a supermartingale.

Note. This result shows in particular that a local martingale null at zero is a supermartingale if it is bounded below by a constant.

Exercise 2.2 *(Equivalent characterisation of arbitrage)* For a finite time horizon $T > 0$, fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ and a semimartingale $S = (S_t)_{0 \leqslant t \leqslant T}$. Recall the following notations:

- Θ_{adm} is the family of admissible integrands for S.
- $G_T(\Theta_{\mathrm{adm}}) := \{G_T(\vartheta) : \vartheta \in \Theta_{\mathrm{adm}}\}$.
- L^0_+ is the family of (equivalence classes, for P-a.s. equality, of) nonnegative random variables.
- (NA) denotes the "general" absence of arbitrage condition

$$G_T(\Theta_{\mathrm{adm}}) \cap L^0_+ = \{0\}.$$

  This says that for any self-financing strategy with zero initial capital, the only way to ensure a nonnegative final value (with probability 1) at expiry is to have value zero at expiry.
- $\mathcal{C}^0_{\mathrm{adm}} := G_T(\Theta_{\mathrm{adm}}) - L^0_+ = \{G_T(\vartheta) - Y : \vartheta \in \Theta_{\mathrm{adm}}, Y \in L^0_+\}$.

Prove that (NA) is equivalent to $\mathcal{C}^0_{\mathrm{adm}} \cap L^\infty \cap L^0_+ = \{0\}$.

Exercise 2.3 *(Equivalent martingale measure)* Let S be a semimartingale with respect to the probability measure P, and suppose $Q \approx P$ is an equivalent probability measure satisfying $E_Q[Y] \leqslant 0$ for all $Y \in \mathcal{C}^0_{\mathrm{adm}} \cap L^\infty$.

(a) Prove that Q satisfies $E_Q[G_T(\vartheta)] \leqslant 0$ for all $\vartheta \in \Theta_{\mathrm{adm}}$.

(b) If S is bounded, prove that S is a Q-martingale.

Exercise 2.4 *(Example of arbitrage on a finite time interval)* Recall from the lecture the arbitrage strategy (with zero initial value) on the infinite time interval $[0, \infty)$ which is given by

$$\vartheta = \mathbf{1}_{]\!]0,\tau]\!]}, \quad \tau := \inf\{t \geqslant 0 : W_t = 1\},$$

where W is a Brownian motion. Note that for the above strategy, we must be on the infinite time interval $[0, \infty)$ because although $\tau < \infty$ a.s., τ is unbounded.

Construct a similar arbitrage strategy on the interval $[0, T]$, where $T > 0$ is a fixed finite horizon.

Hint: Consider the geometric Brownian motion $\overline{S} = (\overline{S}_t)_{t \geqslant 0}$ given by

$$\overline{S}_t = \exp\left(W_t - \frac{1}{2}t\right),$$

which is adapted to the filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \geqslant 0}$. Apply the time-change $t = \tan u$ and let $T = \pi/2$ be the expiry time (after the time change).