

Mathematical Finance

Exercise Sheet 2

Submit by 12:00 on Wednesday, October 4 via the course homepage.

Exercise 2.1 (*From local martingale to supermartingale*) Let $(X_t)_{t \geq 0}$ be a local martingale null at zero and $(Y_t)_{t \geq 0}$ a martingale such that $Y_t \leq X_t$ for each $t \geq 0$. Prove that X is a supermartingale.

Note. This result shows in particular that a local martingale null at zero is a supermartingale if it is bounded below by a constant.

Solution 2.1 Define the difference process $Z := X - Y$. Note that since $X = Z + Y$ and Y is a martingale (and hence a supermartingale), it suffices to show that Z is a supermartingale.

Since a martingale is a local martingale, and the space of local martingales is a vector space, we know that Z is a local martingale. Let (τ_n) be a localising sequence for Z . Since Z is nonnegative, we can apply the Fatou lemma to get

$$E[Z_t | \mathcal{F}_s] \leq \liminf_{n \rightarrow \infty} E[Z_{t \wedge \tau_n} | \mathcal{F}_s] = \liminf_{n \rightarrow \infty} Z_{s \wedge \tau_n} = Z_s.$$

Taking expectations of the above with $s = 0$ gives

$$0 \leq E[Z_t] \leq E[Z_0] = -E[Y_0] < \infty.$$

It follows that Z is integrable and a supermartingale. This completes the proof.

Exercise 2.2 (*Equivalent characterisation of arbitrage*) For a finite time horizon $T > 0$, fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and a semimartingale $S = (S_t)_{0 \leq t \leq T}$. Recall the following notations:

- Θ_{adm} is the family of admissible integrands for S .
- $G_T(\Theta_{\text{adm}}) := \{G_T(\vartheta) : \vartheta \in \Theta_{\text{adm}}\}$.
- L_+^0 is the family of (equivalence classes, for P -a.s. equality, of) nonnegative random variables.
- (NA) denotes the “general” absence of arbitrage condition

$$G_T(\Theta_{\text{adm}}) \cap L_+^0 = \{0\}.$$

This says that for any self-financing strategy with zero initial capital, the only way to ensure a nonnegative final value (with probability 1) at expiry is to have value zero at expiry.

- $\mathcal{C}_{\text{adm}}^0 := G_T(\Theta_{\text{adm}}) - L_+^0 = \{G_T(\vartheta) - Y : \vartheta \in \Theta_{\text{adm}}, Y \in L_+^0\}$.

Prove that (NA) is equivalent to $\mathcal{C}_{\text{adm}}^0 \cap L^\infty \cap L_+^0 = \{0\}$.

Solution 2.2 First assume that (NA) holds. Take $G_T(\vartheta) - Y \in \mathcal{C}_{\text{adm}}^0 \cap L_+^0$. Since $G_T(\vartheta) - Y \in L_+^0$ and $Y \in L_+^0$, then also $G_T(\vartheta) \in L_+^0$, and thus by (NA), $G_T(\vartheta) = 0$. We thus have $G_T(\vartheta) - Y = -Y \in L_+^0$, and hence $Y = 0$, so that $G_T(\vartheta) - Y = 0$. It follows that $\mathcal{C}_{\text{adm}}^0 \cap L^\infty \cap L_+^0 \subseteq \{0\}$, and since clearly $0 \in \mathcal{C}_{\text{adm}}^0 \cap L^\infty \cap L_+^0$, we have equality.

Conversely, take $G_T(\vartheta) \in G_T(\Theta_{\text{adm}}) \cap L_+^0$. We need to show that $G_T(\vartheta) = 0$. Note that for each $n \in \mathbb{N}$,

$$G_T(\vartheta) \wedge n = G_T(\vartheta) - (G_T(\vartheta) - n) \vee 0 \in \mathcal{C}_{\text{adm}}^0.$$

Moreover, $0 \leq G_T(\vartheta) \wedge n \leq n$ so that $G_T(\vartheta) \wedge n \in \mathcal{C}_{\text{adm}}^0 \cap L^\infty \cap L_+^0$, and hence by assumption $G_T(\vartheta) \wedge n = 0$. Finally, we have

$$G_T(\vartheta) = \lim_{n \rightarrow \infty} G_T(\vartheta) \wedge n = 0,$$

as required.

Exercise 2.3 (*Equivalent martingale measure*) Let S be a semimartingale with respect to the probability measure P , and suppose $Q \approx P$ is an equivalent probability measure satisfying $E_Q[Y] \leq 0$ for all $Y \in \mathcal{C}_{\text{adm}}^0 \cap L^\infty$.

- Prove that Q satisfies $E_Q[G_T(\vartheta)] \leq 0$ for all $\vartheta \in \Theta_{\text{adm}}$.
- If S is bounded, prove that S is a Q -martingale.

Solution 2.3

- Let $\vartheta \in \Theta_{\text{adm}}$. For $n \in \mathbb{N}$, note that

$$G_T(\vartheta) \wedge n = G_T(\vartheta) - (G_T(\vartheta) - n) \vee 0 \in \mathcal{C}_{\text{adm}}^0.$$

Since $\vartheta \in \Theta_{\text{adm}}$, there is some $a > 0$ such that $G_T(\vartheta) \geq -a$. Also, we have $G_T(\vartheta) \wedge n \leq n$ by construction, and thus $-a \leq G_T(\vartheta) \wedge n \leq n$ so that $G_T(\vartheta) \wedge n \in L^\infty$. Hence, $G_T(\vartheta) \wedge n \in \mathcal{C}_{\text{adm}}^0 \cap L^\infty$. It follows from the assumption that $E_Q[G_T(\vartheta) \wedge n] \leq 0$. Fatou's lemma thus gives

$$E_Q[G_T(\vartheta)] \leq \liminf_{n \rightarrow \infty} E_Q[G_T(\vartheta) \wedge n] \leq 0,$$

and hence $E_Q[G_T(\vartheta)] \leq 0$, as required.

- (b) We are given that S is a semimartingale (with respect to P) and thus adapted. Also, S is integrable since it is bounded. It remains to verify that S satisfies the martingale property. So fix times $0 \leq s < t$ and $A \in \mathcal{F}_s$. It suffices to show that

$$E_Q[(S_t - S_s)\mathbf{1}_A] = 0, \quad (1)$$

since this implies that $E_Q[S_t - S_s \mid \mathcal{F}_s] = 0$ by the definition of the conditional expectation.

Now, define the process $\vartheta = \mathbf{1}_{A \times (s,t]}$. Then ϑ is predictable, S -integrable and admissible, since S is bounded. We compute

$$G_T(\vartheta) = \int_0^T \vartheta \, dS = \mathbf{1}_A(S_t - S_s).$$

With (1) in mind, it remains to show $E_Q[G_T(\vartheta)] = 0$. By part (a), we have $E_Q[G_T(\vartheta)] \leq 0$. By considering $-\vartheta \in \Theta_{\text{adm}}$, we also have $E_Q[G_T(-\vartheta)] \leq 0$, and since $G_T(-\vartheta) = -G_T(\vartheta)$, this yields $E_Q[G_T(\vartheta)] = 0$, completing the proof.

Exercise 2.4 (*Example of arbitrage on a finite time interval*) Recall from the lecture the arbitrage strategy (with zero initial value) on the infinite time interval $[0, \infty)$ which is given by

$$\vartheta = \mathbf{1}_{]0, \tau]}, \quad \tau := \inf\{t \geq 0 : W_t = 1\},$$

where W is a Brownian motion. Note that for the above strategy, we must be on the infinite time interval $[0, \infty)$ because although $\tau < \infty$ a.s., τ is unbounded.

Construct a similar arbitrage strategy on the interval $[0, T]$, where $T > 0$ is a fixed finite horizon.

Hint: Consider the geometric Brownian motion $\bar{S} = (\bar{S}_t)_{t \geq 0}$ given by

$$\bar{S}_t = \exp\left(W_t - \frac{1}{2}t\right),$$

which is adapted to the filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \geq 0}$. Apply the time-change $t = \tan u$ and let $T = \pi/2$ be the expiry time (after the time change).

Solution 2.4 By the law of the iterated logarithm, we have $\lim_{t \rightarrow \infty} \bar{S}_t = 0$. So the stopping time $\bar{\tau} := \inf\{t \geq 0 : \bar{S}_t = 1/2\}$ is a.s. finite. Define the process $S = (S_u)_{0 \leq u \leq \pi/2}$ by

$$S_u = \bar{S}_{\tan u}, \quad 0 \leq u < \pi/2,$$

and $S_{\pi/2} = 0$. Then S is adapted to the filtration $\mathbb{F} = (\mathcal{F}_u)_{0 \leq u \leq \pi/2}$ defined by $\mathcal{F}_u = \bar{\mathcal{F}}_{\tan u}$. Note also that since $\lim_{t \rightarrow \infty} \bar{S}_t = 0$, the process S is continuous. Define $\tau := \arctan \bar{\tau}$, which is a stopping time with respect to \mathbb{F} (since \tan is strictly

increasing on $[0, \pi/2]$ satisfying $0 \leq \tau < \pi/2$. We define the self-financing strategy $\varphi \hat{=} (0, \vartheta)$ by $\vartheta := -\mathbf{1}_{\llbracket 0, \tau \rrbracket}$ (note ϑ is adapted and left-continuous, thus predictable). We compute

$$V_t(\varphi) = \int_0^t \vartheta_u dS_u = - \int_0^{t \wedge \tau} dS_u = S_0 - S_t^\tau = 1 - S_t^\tau.$$

So since $\tau < \pi/2$, we have $V_{\pi/2}(\varphi) = 1 - S_\tau = 1 - \bar{S}_\tau = 1/2$, and thus $\varphi \hat{=} (0, \vartheta)$ is an arbitrage strategy.