Coordinator
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Mathematical Finance

Exercise Sheet 4

Submit by 12:00 on Wednesday, October 18 via the course homepage.

Exercise 4.1 Fix a finite time horizon $T > 0$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leq T}$, and an adapted RCLL process $S = (S_t)_{0 \leqslant t \leqslant T}$. Recall that b$\mathcal{E}$ denotes the space of "simple integrands" ϑ, i.e., processes of the form

$$\vartheta = \sum_{i=1}^{n} h_i \mathbf{1}_{]\!]\tau_{i-1}, \tau_i]\!]},$$

where $n \in \mathbb{N}$, $0 \leqslant \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant T$ are stopping times, and h_i are $\mathbb{R}^d$-valued, bounded and $\mathcal{F}_{\tau_{i-1}}$-measurable random variables. For $\vartheta \in \mathrm{b}\mathcal{E}$, we have

$$G_t(\vartheta) = \sum_{i=1}^{n} h_i (S_{\tau_i \wedge t} - S_{\tau_{i-1} \wedge t}).$$

In the lecture, we have seen that the existence of an equivalent local martingale measure (ELMM) guarantees $\mathrm{NA}^{\mathrm{adm}}_{\mathrm{elem}}$. However, the example below illustrates that if we remove the admissibility constraint, this result fails; that is, the existence of an ELMM does not guarantee $\mathrm{NA}_{\mathrm{elem}}$ holds. Note that this corrects an erroneous statement from the lecture.

Let $W = (W_t)_{0 \leqslant t \leqslant T}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and define S by

$$S_t := \begin{cases} \frac{1}{\sqrt{2(T-t)}} \exp\left(-\frac{W_t^2}{2(T-t)}\right) & \text{if } 0 \leqslant t < T, \\ 0 & \text{if } t = T. \end{cases}$$

(a) Show that S is a continuous local P-martingale.

(b) Show that S is not a P-martingale.

(c) Construct a *non-admissible* "one-step buy-and-hold" simple strategy which gives an arbitrage opportunity.

Exercise 4.2 Fix a finite time horizon $T > 0$, a semimartingale $S = (S_t)_{0 \leqslant t \leqslant T}$ and a simple integrand $\vartheta \in \mathrm{b}\mathcal{E}$ with

$$\vartheta = \sum_{i=1}^{n} h_i \mathbf{1}_{]\!]\tau_{i-1}, \tau_i]\!]},$$

where $n \in \mathbb{N}$, $0 \leqslant \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant T$ are stopping times, and h_i are $\mathbb{R}^d$-valued, bounded, and $\mathcal{F}_{\tau_{i-1}}$-measurable random variables. Suppose moreover that $G_T(\vartheta) \geqslant 0$ P-a.s.

Consider the following two statements (a) and (b).

(a) Suppose that $G_T(\vartheta) \not\equiv 0$. Then there exists $\vartheta' \in \mathrm{b}\mathcal{E}$ with $\vartheta' = \sum_{k=1}^m h'_k \mathbf{1}_{\rrbracket \tau'_{k-1}, \tau'_k \rrbracket}$ and $G_T(\vartheta') \geqslant 0$ P-a.s. as well as $G_T(\vartheta') \not\equiv 0$ and such that ϑ' is 0-admissible in discrete time, in the sense that $G_{\tau'_k}(\vartheta') \geqslant 0$ P-a.s. for all k.

(b) Suppose S admits an ELMM Q. By using the DMW theorem in the discrete-time model with (random) trading dates $\tau_0, \dots, \tau_n$, we deduce that $G_T(\vartheta) = 0$ P-a.s.

Prove or disprove (a) and (b). If things go wrong, identify where as precisely as possible. What changes in (b) if Q is an EMM?

Exercise 4.3 Suppose that $f, g : [0, T] \to \mathbb{R}$ are functions of finite variation. Establish the integration by parts formulas

$$\begin{aligned} f(T)g(T) - f(0)g(0) &= \int_0^T f(s)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) \\ &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s) \\ &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) + \sum_{0<s\leqslant T} \Delta f(s)\Delta g(s). \end{aligned}$$