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Mathematical Finance

Exercise Sheet 4

Submit by 12:00 on Wednesday, October 18 via the course homepage.

Exercise 4.1 Fix a finite time horizon $T > 0$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leq T}$, and an adapted RCLL process $S = (S_t)_{0 \leqslant t \leqslant T}$. Recall that $\mathrm{b}\mathcal{E}$ denotes the space of "simple integrands" ϑ, i.e., processes of the form

$$\vartheta = \sum_{i=1}^{n} h_i \mathbf{1}_{]\!]\tau_{i-1}, \tau_i]\!]},$$

where $n \in \mathbb{N}$, $0 \leqslant \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant T$ are stopping times, and h_i are $\mathbb{R}^d$-valued, bounded and $\mathcal{F}_{\tau_{i-1}}$-measurable random variables. For $\vartheta \in \mathrm{b}\mathcal{E}$, we have

$$G_t(\vartheta) = \sum_{i=1}^{n} h_i (S_{\tau_i \wedge t} - S_{\tau_{i-1} \wedge t}).$$

In the lecture, we have seen that the existence of an equivalent local martingale measure (ELMM) guarantees $\mathrm{NA}^{\mathrm{adm}}_{\mathrm{elem}}$. However, the example below illustrates that if we remove the admissibility constraint, this result fails; that is, the existence of an ELMM does not guarantee $\mathrm{NA}_{\mathrm{elem}}$ holds. Note that this corrects an erroneous statement from the lecture.

Let $W = (W_t)_{0 \leqslant t \leqslant T}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and define S by

$$S_t := \begin{cases} \frac{1}{\sqrt{2(T-t)}} \exp\left(-\frac{W_t^2}{2(T-t)}\right) & \text{if } 0 \leqslant t < T, \\ 0 & \text{if } t = T. \end{cases}$$

(a) Show that S is a continuous local P-martingale.

(b) Show that S is not a P-martingale.

(c) Construct a *non-admissible* "one-step buy-and-hold" simple strategy which gives an arbitrage opportunity.

Solution 4.1

(a) We first check continuity of S. Clearly, S is continuous on $[0, T)$, and thus it suffices to show $\lim_{t \uparrow T} S_t = S_T = 0$ P-a.s. Consider the nullset $N = \{W_T = 0\}$. Since W is P-a.s. continuous on $[0, T]$, there are for each $\omega \in N^c$ some $K(\omega) > 0$

and $\delta(\omega) > 0$ such that $|W_t(\omega)| \geqslant K(\omega)$ for all $t \in [T - \delta(\omega), T]$. We thus have for each $\omega \in N^c$ that

$$\begin{aligned}\limsup_{t\uparrow T} S_t(\omega) &\leqslant \lim_{t\uparrow T} \frac{1}{\sqrt{2(T-t)}} \exp\left(-\frac{K^2(\omega)}{2(T-t)}\right) = \lim_{t\downarrow 0} \frac{\exp\left(-\frac{K^2(\omega)}{t^2}\right)}{t} \\ &\leqslant \lim_{t\downarrow 0} \frac{1}{t(1+\frac{K^2(\omega)}{t^2})} = 0,\end{aligned}$$

where the second inequality uses $\exp(x) < \frac{1}{1-x}$ for all $x < 1$. Since $S_t(\omega) \geqslant 0$, we have

$$0 \leqslant \liminf_{t\uparrow T} S_t(\omega) \leqslant \limsup_{t\uparrow T} S_t(\omega) \leqslant 0$$

so that $\lim_{t\uparrow T} S_t(\omega) = 0$ as required. We have thus shown that S is P-a.s. continuous.

It remains to show that S is a local martingale. To this end, consider the function $f(t,x) = \frac{1}{\sqrt{2(T-t)}} \exp(-\frac{x^2}{2(T-t)})$. Then $f \in C^2([0,T) \times \mathbb{R}; \mathbb{R})$, and we compute

$$\begin{aligned}\frac{\partial f}{\partial t}(t,x) &= \frac{1}{(2(T-t))^{3/2}} \exp\left(-\frac{x^2}{2(T-t)}\right) \\ &\quad - \frac{1}{\sqrt{2(T-t)}} \exp\left(-\frac{x^2}{2(T-t)}\right) \frac{2x^2}{(2(T-t))^2} \\ &= \left(\frac{1}{(2(T-t))^{3/2}} - \frac{2x^2}{(2(T-t))^{5/2}}\right) \exp\left(-\frac{x^2}{2(T-t)}\right), \\ \frac{\partial f}{\partial x}(t,x) &= -\frac{2x}{(2(T-t))^{3/2}} \exp\left(-\frac{x^2}{2(T-t)}\right), \\ \frac{\partial^2 f}{\partial x^2}(t,x) &= \left(-\frac{2}{(2(T-t))^{3/2}} + \frac{4x^2}{(2(T-t))^{5/2}}\right) \exp\left(-\frac{x^2}{2(T-t)}\right).\end{aligned}$$

We then apply Itô's formula to get for all $t \in [0,T)$ that

$$\mathrm{d}S_t = -\frac{2W_t}{(2(T-t))^{3/2}} \exp\left(-\frac{W_t^2}{2(T-t)}\right) \mathrm{d}W_t.$$

Since the above integrand is continuous and adapted on $[0,T)$, it follows that S is a continuous local martingale on $[0,T)$.

Now for each $n \in \mathbb{N}$, define the stopping time

$$\tau_n := \inf\{t \geqslant 0 : |S_t| \geqslant n\} \wedge T.$$

Since S is continuous on the compact interval $[0,T]$, we have $\tau_n \uparrow T$ stationarily P-a.s. Moreover, the stopped process S^{τ_n} is a bounded local martingale on $[0, T)$, and thus a martingale on $[0, T)$. For $t \in [0, T)$, the dominated convergence theorem yields

$$E[S_T^{\tau_n} \mid \mathcal{F}_t] = \lim_{u \uparrow T} E[S_u^{\tau_n} \mid \mathcal{F}_t] = \lim_{u \uparrow T} S_t^{\tau_n} = S_t^{\tau_n},$$

and $t = T$ gives $E[S_T^{\tau_n} \mid \mathcal{F}_T] = S_T^{\tau_n}$. It follows that S is a continuous local martingale on $[0, T]$ (and that (τ_n) is a localising sequence).

(b) Note that

$$S_0 = \frac{1}{\sqrt{2T}} \qquad \text{and} \qquad S_T = 0,$$

and thus in particular $E[S_0] \neq E[S_T]$, so that S is not a martingale.

(c) Take $\vartheta := -\mathbf{1}_{]\!]0,T]\!]}$. Then for all $t \in [0, T]$,

$$G_t(\vartheta) = S_0 - S_t = \frac{1}{\sqrt{2T}} - S_t.$$

If ϑ was admissible, there would be some $a > 0$ such that $\frac{1}{\sqrt{2T}} - S_t \geqslant -a$ for all $t \in [0, T]$, and then

$$0 \leqslant S_t \leqslant \frac{1}{\sqrt{2T}} + a.$$

By Exercise 2.1, it would follow that S is a martingale, which contradicts part (b). Thus ϑ is not admissible. However, $G_T(\vartheta) = \frac{1}{\sqrt{2T}} > 0$ since $S_T = 0$, and thus ϑ induces a simple arbitrage strategy. This completes the proof.

Exercise 4.2 Fix a finite time horizon $T > 0$, a semimartingale $S = (S_t)_{0 \leqslant t \leqslant T}$ and a simple integrand $\vartheta \in \mathrm{b}\mathcal{E}$ with

$$\vartheta = \sum_{i=1}^{n} h_i \mathbf{1}_{]\!]\tau_{i-1}, \tau_i]\!]},$$

where $n \in \mathbb{N}$, $0 \leqslant \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant T$ are stopping times, and h_i are $\mathbb{R}^d$-valued, bounded, and $\mathcal{F}_{\tau_{i-1}}$-measurable random variables. Suppose moreover that $G_T(\vartheta) \geqslant 0$ P-a.s.

Consider the following two statements (a) and (b).

(a) Suppose that $G_T(\vartheta) \not\equiv 0$. Then there exists $\vartheta' \in \mathrm{b}\mathcal{E}$ with $\vartheta' = \sum_{k=1}^{m} h'_k \mathbf{1}_{]\!]\tau'_{k-1}, \tau'_k]\!]}$ and $G_T(\vartheta') \geqslant 0$ P-a.s. as well as $G_T(\vartheta') \not\equiv 0$ and such that ϑ' is 0-admissible in discrete time, in the sense that $G_{\tau'_k}(\vartheta') \geqslant 0$ P-a.s. for all k.

(b) Suppose S admits an ELMM Q. By using the DMW theorem in the discrete-time model with (random) trading dates $\tau_0, \ldots, \tau_n$, we deduce that $G_T(\vartheta) = 0$ P-a.s.

Prove or disprove (a) and (b). If things go wrong, identify where as precisely as possible. What changes in (b) if Q is an EMM?

Solution 4.2

(a) This statement is true.

If $P[G_{\tau_i} < 0] = 0$ for all i, then take $\vartheta' \equiv \vartheta$. Otherwise, let $k_0 \in \mathbb{N}$ be the largest integer with $P[G_{\tau_{k_0}} < 0] > 0$, i.e.

$$k_0 := \max\{i = 0, \dots, n : P[G_{\tau_i} < 0] > 0\},$$

and define the event

$$A := \{G_{\tau_{k_0}}(\vartheta) < 0\} \in \mathcal{F}_{\tau_{k_0}}.$$

Note that $0 < \tau_{k_0} < T$, since $G_0(\vartheta) = 0$ and $G_T(\vartheta) \geqslant 0$ P-a.s. We define ϑ' so that the corresponding self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$ is to wait until time τ_{k_0}, and then to follow $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ on A, i.e.,

$$\vartheta'_t := \begin{cases} 0 & \text{if } t \leqslant \tau_{k_0}, \\ \vartheta_t \mathbf{1}_A & \text{if } t > \tau_{k_0}. \end{cases}$$

Then $\vartheta' \in \mathrm{b}\mathcal{E}$. Moreover, for $k \geqslant k_0$ we have

$$G_{\tau_k}(\vartheta') = \mathbf{1}_A\Big(G_{\tau_k}(\vartheta) - G_{\tau_{k_0}}(\vartheta)\Big).$$

By definition of A we have $G_{\tau_k}(\vartheta') \geqslant 0$ P-a.s., and thus ϑ' is 0-admissible in discrete time. Finally, on A we have $G_T(\vartheta') > 0$, so that $G_T(\vartheta') \not\equiv 0$. This completes the proof.

(b) This statement is false.

The DMW theorem asserts that if we are in discrete time and there exists an ELMM for S, then we have no arbitrage. We are given that S admits a continuous-time ELMM, and thus a natural way to try to (naively) prove statement (b) is to attempt to construct a discrete-time ELMM from the given continuous-time ELMM Q. Exercise 4.1 shows that this cannot be done in general.

Now, if we are given that Q is an EMM for S, then we have

$$\begin{aligned} E_Q[G_T(\vartheta)] &= \sum_{i=1}^n E_Q[h_i(S_{\tau_i} - S_{\tau_{i-1}})] = E_Q\Big[E_Q[h_i(S_{\tau_i} - S_{\tau_{i-1}}) \mid \mathcal{F}_{\tau_{i-1}}]\Big] \\ &= E\Big[h_i E_Q[S_{\tau_i} - S_{\tau_{i-1}} \mid \mathcal{F}_{\tau_{i-1}}]\Big] = 0 \end{aligned}$$

since S is a Q-martingale and h_i is bounded and $\mathcal{F}_{\tau_i - 1}$-measurable. We also know that $G_T(\vartheta) \geqslant 0$ P-a.s., and since $Q \approx P$, then also $G_T(\vartheta) \geqslant 0$ Q-a.s. Combining this with the equality $E_Q[G_T(\vartheta)] = 0$ gives $G_T(\vartheta) = 0$ Q-a.s., and hence also $G_T(\vartheta) = 0$ P-a.s.

Exercise 4.3 Suppose that $f, g : [0,T] \to \mathbb{R}$ are functions of finite variation. Establish the integration by parts formulas

$$\begin{aligned} f(T)g(T) - f(0)g(0) &= \int_0^T f(s)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) \\ &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s) \\ &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) + \sum_{0<s\leqslant T} \Delta f(s)\Delta g(s). \end{aligned}$$

Solution 4.3 Let μ and ν be the Riemann–Stieljes measures associated to f and g, respectively. That is, for $0 \leqslant s < t \leqslant T$,

$$\mu\big((s,t]\big) := f(t) - f(s),$$

and $\mu(\{0\}) := 0$ (and similarly for ν). It follows by the Fubini–Tonelli theorem that

$$\begin{aligned} \big(f(T) - f(0)\big)\big(g(T) - g(0)\big) &= (\mu \times \nu)\big([0,T] \times [0,T]\big) \\ &= \int_{[0,T]\times[0,T]} \mathrm{d}(\mu \times \nu) = \int_0^T \int_0^T \mathrm{d}\mu(r)\,\mathrm{d}\nu(s) \\ &= \int_0^T \int_0^T \mathbf{1}_{[0,r)}(s)\,\mathrm{d}\mu(r)\,\mathrm{d}\nu(s) \\ &\quad + \int_0^T \int_0^T \mathbf{1}_{[r,T]}(s)\,\mathrm{d}\mu(r)\,\mathrm{d}\nu(s) \\ &= \int_0^T \int_0^T \mathbf{1}_{[0,r)}(s)\,\mathrm{d}\nu(s)\,\mathrm{d}\mu(r) \\ &\quad + \int_0^T \int_0^T \mathbf{1}_{[0,s]}(r)\,\mathrm{d}\mu(r)\,\mathrm{d}\nu(s) \\ &= \int_0^T \nu\big([0,r)\big)\,\mathrm{d}\mu(r) + \int_0^T \mu\big([0,s]\big)\,\mathrm{d}\nu(s) \\ &= \int_0^T \big(g(r-) - g(0)\big)\,\mathrm{d}\mu(r) + \int_0^T \big(f(s) - f(0)\big)\,\mathrm{d}\nu(s) \\ &= \int_0^T g(r-)\,\mathrm{d}f(r) - g(0)\big(f(T) - f(0)\big) \\ &\quad + \int_0^T f(s)\,\mathrm{d}g(s) - f(0)\big(g(T) - g(0)\big). \end{aligned}$$

Rearranging gives

$$f(T)g(T) - f(0)g(0) = \int_0^T f(s)\,\mathrm{d}g(s) + \int_0^T g(r-)\,\mathrm{d}f(r),$$

which is the first equality. By exchanging the roles of f and g, we get the second equality. Now for the last equality, we can write

$$\begin{aligned} f(T)g(T) - f(0)g(0) &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s) \\ &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T \big(g(s-) + \Delta g(s)\big)\,\mathrm{d}f(s) \\ &= \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) + \int_0^T \Delta g(s)\,\mathrm{d}f(s). \end{aligned}$$

It thus suffices to show that

$$\int_0^T \Delta g(s)\,\mathrm{d}f(s) = \sum_{0<s\leqslant T} \Delta f(s)\Delta g(s). \tag{1}$$

Let D_1 and D_2 be the discontinuity points of f and g, respectively. Since f and g are functions of finite variation on $[0, T]$, D_1 and D_2 are at most countable. In particular, the sum in (1) is well-defined as

$$\sum_{0<s\leqslant T} \Delta f(s)\Delta g(s) := \sum_{s\in D_1\cap D_2} \Delta f(s)\Delta g(s).$$

Moreover, note that for each $s \in [0, T]$, $\Delta g(s) \neq 0$ only if $s \in D_2$, and hence

$$\begin{aligned} \int_0^T \Delta g(s)\,\mathrm{d}f(s) &= \int_{D_2} \Delta g(s)\,\mathrm{d}f(s) = \sum_{s\in D_2} \Delta f(s)\Delta g(s) = \sum_{s\in D_1\cap D_2} \Delta f(s)\Delta g(s) \\ &= \sum_{0<s\leqslant T} \Delta f(s)\Delta g(s). \end{aligned}$$

This completes the proof.