Coordinator
Benjamin Kotlov

Mathematical Finance

Exercise Sheet 5

Submit by 12:00 on Wednesday, November 1 via the course homepage.

Exercise 5.1 *(Convergence in probability)* Consider the metric d on L^0 defined by $d(X,Y) := E[1 \wedge |X-Y|]$. Show that for $X_n, X \in L^0$, we have

$$X_n \to X \text{ in probability} \quad \iff \quad d(X_n, X) \to 0.$$

Solution 5.1 First suppose $X_n \to X$ in probability, and fix $\varepsilon \in (0,1)$. We have

$$\begin{aligned} d(X_n, X) &= E[1 \wedge |X_n - X|] \\ &= E[(1 \wedge |X_n - X|)\mathbf{1}_{\{|X_n - X| > \varepsilon\}}] + E[(1 \wedge |X_n - X|)\mathbf{1}_{\{|X_n - X| \leqslant \varepsilon\}}] \\ &\leqslant P[|X_n - X| > \varepsilon] + \varepsilon \to \varepsilon \quad \text{as } n \to \infty, \end{aligned}$$

where in the last step we use that $X_n \to X$ in probability. We have thus shown that for all $\varepsilon \in (0,1)$, $\limsup_{n\to\infty} d(X_n, X) \leqslant \varepsilon$, and hence $\lim_{n\to\infty} d(X_n, X) = 0$.

Conversely, suppose $\lim_{n\to\infty} d(X_n, X) = 0$, and fix $\varepsilon \in (0,1)$. We have

$$\begin{aligned} P[|X_n - X| > \varepsilon] &= P[1 \wedge |X_n - X| > \varepsilon] \\ &\leqslant \varepsilon^{-1} E[1 \wedge |X_n - X|] \\ &= \varepsilon^{-1} d(X_n, X) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This completes the proof.

Exercise 5.2 *(Good integrator)* Fix a finite time horizon $T > 0$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$, and an adapted RCLL process $X = (X_t)_{0 \leqslant t \leqslant T}$. Show that X is a good integrator if and only if the set

$$\mathfrak{X}_{(1)} := \{H \bullet X_T : H \in \mathrm{b}\mathcal{E}, \|H\|_\infty \leqslant 1\}$$

is bounded in L^0, in the sense that $\lim_{n\to\infty} \sup_{Y \in \mathfrak{X}_{(1)}} P[|Y| \geqslant n] = 0$.

Recall that X is a good integrator if whenever $H^n, H \in \mathrm{b}\mathcal{E}$ with $H^n \to H$ uniformly in (ω, t), we have $H^n \bullet X_T \to H \bullet X_T$ in L^0.

Solution 5.2 Assume first that X is a good integrator, and suppose for contradiction that $\mathfrak{X}_{(1)}$ is not bounded in L^0. This means that there is $\varepsilon > 0$ and a sequence

$(H^n) \subseteq \mathrm{b}\mathcal{E}$ such that $\|H^n\|_\infty \leqslant 1$ and $P[|H^n \bullet X_T| \geqslant n] \geqslant \varepsilon$. Then $(\frac{1}{n}H^n) \subseteq \mathrm{b}\mathcal{E}$ with $\|H^n\|_\infty \leqslant \frac{1}{n}$ and

$$P[|\tfrac{1}{n}H^n \bullet X_T| \geqslant 1] = P[|H^n \bullet X_T| \geqslant n] \geqslant \varepsilon,$$

so that in particular $\frac{1}{n}H^n \bullet X_T \not\to 0$ in L^0. This contradicts the assumption that X is a good integrator, since $\|H^n\| \leqslant \frac{1}{n}$ implies that $H^n \to 0$ uniformly in (ω, t). It follows that $\mathfrak{X}_{(1)}$ is bounded in L^0.

Conversely, assume that $\mathfrak{X}_{(1)}$ is bounded in L^0, and suppose $H^n, H \in \mathrm{b}\mathcal{E}$ with $H^n \to H$ uniformly in (ω, t). We need to show that $H^n \bullet X_T \to H \bullet X_T$ in L^0, i.e. that for fixed $\varepsilon > 0$,

$$P[|(H^n - H) \bullet X_T| \geqslant \varepsilon] \to 0 \quad \text{as } n \to \infty. \tag{1}$$

If there exists $N \in \mathbb{N}$ such that $\|H^n - H\|_\infty = 0$ for all $n \geqslant N$, then we also have $(H^n - H) \bullet X_T = 0$ for all $n \geqslant N$ and thus (1) holds trivially.

If there does not exist such $N \in \mathbb{N}$, then we may assume without loss of generality that $\|H^n - H\|_\infty > 0$ for all $n \in \mathbb{N}$. We can then write

$$\begin{aligned} P[|(H^n - H) \bullet X_T| \geqslant \varepsilon] &= P\left[\left|\frac{H^n - H}{\|H^n - H\|_\infty} \bullet X_T\right| \geqslant \frac{\varepsilon}{\|H^n - H\|_\infty}\right] \\ &\leqslant \sup_{\substack{G \in \mathrm{b}\mathcal{E} \\ \|G\| \leqslant 1}} P\left[|G \bullet X_T| \geqslant \frac{\varepsilon}{\|H^n - H\|_\infty}\right]. \end{aligned}$$

Since $\|H^n - H\|_\infty \to 0$ as $n \to \infty$, then $\frac{\varepsilon}{\|H^n - H\|_\infty} \to \infty$ as $n \to \infty$, and thus the right hand side of the above inequality converges to zero as $n \to \infty$, by boundedness of $\mathfrak{X}_{(1)}$. We have thus shown that (1) holds, completing the proof.

Exercise 5.3 *(The spaces $\mathbb{L}$ and $\mathbb{D}$)* Fix a finite time horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ is assumed to be complete. Let $\mathbb{L}$ and $\mathbb{D}$ denote the spaces of adapted LCRL and adapted RCLL processes, respectively. Define the metric

$$d(X^1, X^2) := E[1 \wedge (X^1 - X^2)^*_T] := E\Big[1 \wedge \sup_{0 \leqslant s \leqslant T} |X^1_s - X^2_s|\Big]$$

on both $\mathbb{L}$ and $\mathbb{D}$ (note that convergence with respect to d is exactly uniform (in t) convergence in probability). Show that when equipped with d, both $\mathbb{L}$ and $\mathbb{D}$ are complete metric spaces.

Hint: You may use that the space of (deterministic) LCRL (respectively RCLL) functionals on $[0, T]$ equipped with the supremum norm is a Banach space.

Solution 5.3 As the proof that $(\mathbb{D}, d)$ is a complete metric space is analogous, we only show that $(\mathbb{L}, d)$ is a complete metric space. It is clear from the definition that d is positive and symmetric (and maps $\mathbb{L} \times \mathbb{L}$ into $[0, \infty)$). To see that d satisfies the triangle inequality and is thus a metric, we consider $X^1, X^2, X^3 \in \mathbb{L}$ and compute

$$\begin{aligned} d(X^1, X^3) &= E\Big[1 \wedge \sup_{0\leqslant s\leqslant T} |X_s^1 - X_s^3|\Big] \\ &\leqslant E\Big[1 \wedge \Big(\sup_{0\leqslant s\leqslant T} |X_s^1 - X_s^2| + \sup_{0\leqslant s\leqslant T} |X_s^2 - X_s^3|\Big)\Big] \\ &\leqslant E\Big[1 \wedge \sup_{0\leqslant s\leqslant T} |X_s^1 - X_s^2| + 1 \wedge \sup_{0\leqslant s\leqslant T} |X_s^2 - X_s^3|\Big] \\ &= d(X^1, X^2) + d(X^2, X_3). \end{aligned}$$

It remains to show $(\mathbb{L}, d)$ is complete. To this end, take a Cauchy sequence $(X^n)_{n\in\mathbb{N}} \subseteq \mathbb{L}$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$d(X^{n_k}, X^{n_{k+1}}) = E\Big[1 \wedge \sup_{0\leqslant s\leqslant T} |X^{n_k+1} - X^{n_k}|\Big] \leqslant \frac{1}{2^{2k}}.$$

In particular,

$$\begin{aligned} P[(X^{n_{k+1}} - X^{n_k})_T^* > 2^{-k}] &= P[1 \wedge (X^{n_{k+1}} - X^{n_k})_T^* > 2^{-k}] \\ &\leqslant 2^k E[1 \wedge (X^{n_{k+1}} - X^{n_k})_T^*] \\ &\leqslant \frac{1}{2^k}. \end{aligned}$$

Hence

$$\sum_{k=1}^{\infty} P[(X^{n_{k+1}} - X^{n_k})_T^* > 2^{-k}] < \infty.$$

The Borel–Cantelli lemma implies that

$$P\left[\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{(X^{n_{k+1}} - X^{n_k})_T^* > 2^{-k}\}\right] = 0,$$

and so

$$P\left[\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{(X^{n_{k+1}} - X^{n_k})_T^* \leqslant 2^{-k}\}\right] = 1.$$

Thus with probability 1, $(X^{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to uniform convergence on $[0, T]$. Since the space of (deterministic) LCRL functionals on $[0, T]$ is a Banach space when equipped with the supremum norm, it follows that for almost every $\omega \in \Omega$, there is a (deterministic) LCRL functional $X(\omega) : [0, T] \to \mathbb{R}$ such that $X^{n_k}(\omega) \to X(\omega)$ uniformly on $[0, T]$. For all other ω, define $X(\omega) \equiv 0$. This then defines a stochastic process $X = (X_t)_{0\leqslant t\leqslant T}$ whose sample paths are LCRL and with $X^{n_k} \to X$ uniformly on $[0, T]$, almost surely. In particular, we have for each $t \in [0, T]$ that $X_t = \lim_{k\to\infty} X_t^{n_k}$ almost surely, and thus X_t is $\mathcal{F}_t$-measurable since

$\mathbb{F}$ is complete. We have thus shown that X is adapted and hence $X \in \mathbb{L}$. Since $X^{n_k} \to X$ uniformly on $[0,T]$ with probability 1, the dominated convergence theorem yields $d(X^{n_k}, X) \to 0$ as $k \to \infty$. Finally, using the fact that a Cauchy sequence converges to a limit if and only if it has a subsequence that converges to the same limit, we conclude that $d(X^n, X) \to 0$ as $n \to \infty$. This completes the proof.

Exercise 5.4 *(Stopped good integrator)* Show that a stopped good integrator is a good integrator. That is, if $X = (X_t)_{0\leqslant t\leqslant T}$ is a good integrator and τ is a stopping time, show that X^τ is a good integrator.

Solution 5.4 Take $H^n, H \in \mathrm{b}\mathcal{E}$ with $H^n \to H$ uniformly in (ω, t). We need to show that $(H^n \bullet X^\tau)_T \to (H \bullet X^\tau)_T$ in L^0. Note that

$$(H \bullet X^\tau)_T = (H \bullet X)_\tau = (H\mathbf{1}_{[\![0,\tau]\!]} \bullet X)_T,$$
$$(H^n \bullet X^\tau)_T = (H^n \bullet X)_\tau = (H^n\mathbf{1}_{[\![0,\tau]\!]} \bullet X)_T.$$

It is clear that $H^n\mathbf{1}_{[\![0,\tau]\!]}, H\mathbf{1}_{[\![0,\tau]\!]} \in \mathrm{b}\mathcal{E}$ with $H^n\mathbf{1}_{[\![0,\tau]\!]} \to H\mathbf{1}_{[\![0,\tau]\!]}$ uniformly in (ω, t), and thus since X is a good integrator, we have $(H^n\mathbf{1}_{[\![0,\tau]\!]} \bullet X)_T \to (H\mathbf{1}_{[\![0,\tau]\!]} \bullet X)_T$ in L^0. This completes the proof.

Exercise 5.5 *(Corollary 3.8)* Let $\mathcal{M}_{0,\mathrm{loc}}$ denote the space of local martingales null at zero. Show that if $M \in \mathcal{M}_{0,\mathrm{loc}}$ then $[M]^{1/2}$ is locally integrable.

Solution 5.5 Recall the space $\mathcal{H}^1$ given by

$$\mathcal{H}^1 := \left\{ M = (M_t)_{0\leqslant t\leqslant T} : M \text{ RCLL martingale, } M_T^* := \sup_{0\leqslant t\leqslant T} |M_t| \in L^1 \right\}.$$

By Exercise 3.1, we know that every local martingale is locally in $\mathcal{H}^1$. That is, for each local martingale M, there is a sequence of stopping times $\tau_n \uparrow T$ stationarily such that $M^{\tau_n} \in \mathcal{H}^1$ for all $n \in \mathbb{N}$. So fix $M \in \mathcal{M}_{0,\mathrm{loc}}$ and let (τ_n) be such a sequence. It suffices to show that for each $n \in \mathbb{N}$, $([M]^{\tau_n})^{1/2}$ is integrable, i.e. for all $t \in [0,T]$, $[M]_{t\wedge\tau_n}^{1/2} \in L^1$. Since $[M]$ is increasing (by Lemma 3.6), it suffices to show that $[M]_{\tau_n}^{1/2} \in L^1$. Since $[M]^{\tau_n} = [M^{\tau_n}]$, we have

$$E\Big[[M]_{\tau_n}^{1/2}\Big] = E\Big[[M^{\tau_n}]_T^{1/2}\Big] \leqslant CE\Big[(M^{\tau_n})_T^*\Big],$$

where $C > 0$ is given by Davis' inequality. Since $M^{\tau_n} \in \mathcal{H}^1$, we have $(M^{\tau_n})_T^* \in L^1$, and thus the right-hand side of the above inequality is finite. This completes the proof.

Exercise 5.6 *(Approximation)* Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F}$ is right-continuous, and let S be a semimartingale. Show in detail that for every

$H \in \mathbb{L}$, one can find a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathrm{b}\mathcal{E}$ with $d'_E(H^n \bullet S, H \bullet S) \to 0$ as $n \to \infty$.

Solution 5.6 We first show that $\mathrm{b}\mathcal{E}_0$ is dense in $(\mathbb{L}, d)$.

Fix $H \in \mathbb{L}$ and $\varepsilon > 0$. Assume first that H is bounded. We want to construct some approximation H^ε of H that is constant on the stochastic intervals $]\!]\tau_n, \tau_{n+1}]\!]$, where $\tau_n \uparrow T$ stationarily are some stopping times. To this end, we first attempt to define $\tau'_0 := 0$ and

$$\tau'_{n+1} := \inf\{t > \tau'_n : |H_t - H_{\tau'_n}| \geqslant \varepsilon\} \wedge T,$$

for all $n \geqslant 0$. The problem with the above definition is that the sample paths of H are left-continuous, so if there is a (big) jump at time τ_n, $|H_t - H_{\tau'_n}| \geqslant \varepsilon$ for all t just after τ_n, and thus $\tau'_{n+1} = \tau'_n$, so that $\tau_n \not\to T$.

To get around this, we consider the process $Y = (Y_t)_{0\leqslant t\leqslant T}$ given by

$$Y_t := H_{t+} := \lim_{s\downarrow t} H_s.$$

Then Y is RCLL, and since the filtration $\mathbb{F}$ is right-continuous, Y is adapted, so that $Y \in \mathbb{D}$. We then instead define $\tau_0 := 0$, and for all $n \geqslant 0$,

$$\tau_{n+1} := \inf\{t > \tau_n : |Y_t - Y_{\tau_n}| \geqslant \varepsilon\} \wedge T.$$

Since $Y \in \mathbb{D}$, each τ_n is a stopping time and $\tau_n \uparrow T$ stationarily. Now define

$$H^\varepsilon := H_0 \mathbf{1}_{[\![0]\!]} + \sum_{n=1}^{\infty} Y_{\tau_n} \mathbf{1}_{]\!]\tau_n,\tau_{n+1}]\!]}.$$

We see that $\|H^\varepsilon - H\|_\infty \leqslant \varepsilon$, because $|H^\varepsilon - H| \leqslant \varepsilon$ on $]\!]\tau_n, \tau_{n+1}[\![$ and

$$|H^\varepsilon_{\tau_{n+1}} - H_{\tau_{n+1}}| = |Y_{\tau_n} - H_{\tau_{n+1}-}| \leqslant \varepsilon,$$

by minimality of τ_{n+1}. In particular,

$$d(H^\varepsilon, H) = E\Big[1 \wedge \sup_{0\leqslant s\leqslant T} |H^\varepsilon_s - H_s|\Big] \leqslant \varepsilon.$$

Now define

$$H^{\varepsilon,m} := H_0 \mathbf{1}_{[\![0]\!]} + \sum_{n=1}^{m} Y_{\tau_n} \mathbf{1}_{]\!]\tau_n,\tau_{n+1}]\!]} \in \mathrm{b}\mathcal{E}_0.$$

Note here we use boundedness of H (so that also Y is bounded, by the same bound) to conclude $H^{\varepsilon,m} \in \mathrm{b}\mathcal{E}_0$. As $\tau_n \uparrow T$ stationarily, then with probability 1, we eventually have $H^{\varepsilon,m} = H^\varepsilon$, so that in particular

$$\lim_{m\to\infty} \sup_{0\leqslant s\leqslant T} |H^{\varepsilon,m}_s - H^\varepsilon_s| = 0, \quad \text{a.s.}$$

The dominated convergence theorem thus yields

$$d(H^{\varepsilon,m}, H^{\varepsilon}) = E\Big[1 \wedge \sup_{0\leqslant s\leqslant T} |H^{\varepsilon,m}_s - H^{\varepsilon}_s|\Big] \to 0,$$

and thus there exists $m(\varepsilon) \in \mathbb{N}$ such that $d(H^{\varepsilon,m(\varepsilon)}, H^{\varepsilon}) < \varepsilon$. The triangle inequality thus gives

$$d(H^{\varepsilon,m(\varepsilon)}, H) \leqslant 2\varepsilon.$$

We have thus shown that when H is bounded, we can approximate it in $(\mathbb{L}, d)$ by elements of $\mathrm{b}\mathcal{E}_0$.

Now consider the general case when H is unbounded. Fix some $\delta > 0$. For each $n \in \mathbb{N}$ define the stopping time $\sigma_n := \inf\{t \geqslant 0 : |H_t| > n\} \wedge T$. Then $\sigma_n \uparrow T$ stationarily, and $H^{\sigma_n} \in \mathbb{L}$ is bounded. So for fixed $\varepsilon > 0$ and every $n \in \mathbb{N}$ we can find some $K^n \in \mathrm{b}\mathcal{E}_0$ with $P[(H^{\sigma_n} - K^n)^*_T > \delta] \leqslant \varepsilon$. We then have

$$\begin{aligned} P[(H - K^n)^*_T > \delta] &\leqslant P[\sigma_n < T] + P[(H^{\sigma_n} - K^n)^*_T > \delta] \\ &\leqslant P[\sigma_n < T] + \varepsilon. \end{aligned}$$

Since $\sigma_n \uparrow T$ stationarily, there is some $N \in \mathbb{N}$ such that for all $n \geqslant N$, $P[\sigma_n < T] \leqslant \varepsilon$, and hence $P[(H - K^n)^*_T > \delta] \leqslant 2\varepsilon$ for all $n \geqslant N$. This shows that $K^n \to H$ in $(\mathbb{L}, d)$, and thus $\mathrm{b}\mathcal{E}_0$ is dense in $(\mathbb{L}, d)$, as claimed.

Alternatively, we could consider $H^n := -n \vee H \wedge n \in \mathbb{L}$, which is bounded, and note that since $H \in \mathbb{L}$, it is almost-surely bounded on the compact interval $[0, T]$ (with a different bound for each ω), and thus

$$\lim_{n\to\infty} \sup_{0\leqslant s\leqslant T} |H^n_s - H_s| = 0, \quad a.s.$$

The dominated convergence theorem then gives $d(H^n, H) \to 0$ as $n \to \infty$. Since we can approximate each H^n by elements of $\mathrm{b}\mathcal{E}_0$, the same holds for H, which again gives us the claim.

Now that we have established that $\mathrm{b}\mathcal{E}_0 \subseteq (\mathbb{L}, d)$ is dense, we take a sequence $(K^n) \subseteq \mathrm{b}\mathcal{E}_0$ with $d(K^n, H) \to 0$. We know from Theorem 3.5 that the integration map

$$J_S : (\mathbb{L}, d) \to (\mathcal{S}, d'_E)$$

is continuous. It thus follows that $d'_E(K^n \bullet S, H \bullet S) \to 0$. Now for each $n \in \mathbb{N}$ we set $H^n := K^n \mathbf{1}_{]\!]0,T]\!]} \in \mathrm{b}\mathcal{E}$. Note H^n is the same as K^n except that at time 0, $H^n_0 = 0$. It follows that $K^n \bullet S = H^n \bullet S$, and thus we also have $d'_E(H^n \bullet S, H \bullet S) \to 0$. This completes the proof.