Coordinator
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Mathematical Finance

Exercise Sheet 6

Submit by 12:00 on Wednesday, November 8 via the course homepage.

Exercise 6.1 *(Bounded in L^0)* Show that a nonempty set $C \subseteq L^0$ is bounded in L^0 if and only if for every sequence $(X_n)_{n\in\mathbb{N}} \subseteq C$ and every sequence of scalars $\lambda_n \to 0$, we have $\lambda_n X_n \to 0$ in L^0.

Solution 6.1 Suppose $C \subseteq L^0$ is bounded so that $\lim_{n\to\infty} \sup_{X\in C} P[|X| > n] = 0$, and fix a subsequence $(X_n)_{n\in\mathbb{N}} \subseteq C$ and a sequence of scalars $\lambda_n \to 0$, where we may assume $\lambda_n \neq 0$ for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$. We need to show that

$$P[|\lambda_n X_n| > \varepsilon] = P[|X_n| > \varepsilon/|\lambda_n|] \to 0 \quad \text{as } n \to \infty.$$

To this end, we fix $\delta > 0$, and using the boundedness of C, we choose some $n_0 \in \mathbb{N}$ such that $\sup_{X\in C} P[|X| > n_0] \leqslant \delta$. Since $\lambda_n \to 0$, there is $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $\varepsilon/|\lambda_n| > n_0$. So then for all $n \geqslant N$,

$$P[|\lambda_n X_n| > \varepsilon] \leqslant P[|X_n| > n_0] \leqslant \sup_{X\in C} P[|X_n| > n_0] \leqslant \delta.$$

As $\delta > 0$ was arbitrary, this implies $\lambda_n X_n \to 0$ in L^0.

Conversely, suppose that for any sequence $(X_n)_{n\in\mathbb{N}} \subseteq C$ and any sequence of scalars $\lambda_n \to 0$, we have $\lambda_n X_n \to 0$ in L^0. Suppose for a contradiction that C is not bounded in L^0. Then there is some $\delta > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_{X\in C} P[|X| > n] \geqslant 2\delta.$$

In particular, by the definition of the supremum, we can find a sequence $(X_n) \subseteq C$ such that

$$P[|X_n| > n] \geqslant \delta, \quad \forall n \in \mathbb{N}.$$

But then

$$P\Big[|\tfrac{1}{n} X_n| > 1\Big] \geqslant \delta, \quad \forall n \in \mathbb{N},$$

so that $\frac{1}{n} X_n \not\to 0$ in L^0. This contradicts our assumption and thus completes the proof.

Exercise 6.2 *(Quadratic covariation)* Recall that for a semimartingale S, the *optional quadratic variation* process is given by

$$[S] := S^2 - S_0^2 - 2\int S_- \, \mathrm{d}S.$$

For two semimartingales X and Y, we define the *optional quadratic covariation* process to be

$$[X, Y] := \frac{1}{4}([X+Y] - [X-Y]).$$

Note that this definition is "consistent" with the optional quadratic variation in the sense that $[X, X] = [X]$.

(a) Establish the integration by parts formula

$$XY = X_0Y_0 + \int X_- \, \mathrm{d}Y + \int Y_- \, \mathrm{d}X + [X, Y].$$

(b) Show that $\Delta[X, Y] = \Delta X \Delta Y$.

(c) Show that $\sum_{0<t\leqslant T}(\Delta X_t)^2 \leqslant [X]_T$.

In particular, $\sum_{0<t\leqslant T}(\Delta X_t)^2$ is P-a.s. convergent (while $\sum_{0<t\leqslant T} |\Delta X_t|$ need not converge).

Solution 6.2

(a) Using the definition of quadratic covariation, we write

$$\begin{aligned}
4[X, Y] &= [X+Y] - [X-Y] \\
&= (X+Y)^2 - (X+Y)_0^2 - 2\int (X+Y)_- \, \mathrm{d}(X+Y) \\
&\quad - (X-Y)^2 + (X-Y)_0^2 + 2\int (X-Y)_- \, \mathrm{d}(X-Y) \\
&= 4XY - 4X_0Y_0 - 4\int X_- \, \mathrm{d}Y - 4\int Y_- \, \mathrm{d}X.
\end{aligned}$$

This rearranges to the integration by parts formula, completing the proof.

(b) Using the integration by parts formula, we have

$$\begin{aligned}
\Delta[X, Y] &= \Delta(XY) - X_-\Delta Y - Y_-\Delta X \\
&= XY - X_-Y_- - X_-(Y - Y_-) - Y_-(X - X_-) \\
&= XY - X_-Y - XY_- + X_-Y_- \\
&= (X - X_-)(Y - Y_-) \\
&= \Delta X \Delta Y.
\end{aligned}$$

(c) By part (b), we have

$$\sum_{0<t\leqslant T}(\Delta X_t)^2 = \sum_{0<t\leqslant T}\Delta[X,X]_t = \sum_{0<t\leqslant T}\Delta[X]_t.$$

Since the map $t \mapsto [X]_t$ is increasing and $[X]_0 = 0$, we have

$$\sum_{0<t\leqslant T}\Delta[X]_t \leqslant [X]_T < \infty,$$

as required.

Exercise 6.3 *(Semimartingales)* Show that $X \in \mathbb{D}$ is a semimartingale if and only if $d'_E(\lambda_n X, 0) \to 0$ whenever $\lambda_n \to 0$ in $\mathbb{R}$.

Solution 6.3 Assume first that $X \in \mathbb{D}$ is a semimartingale, and let $\lambda_n \to 0$ in $\mathbb{R}$. We need to show that $d'_E(\lambda_n X, 0) \to 0$. Recall by Lemma 3.2 that this is equivalent to showing that $H^n \bullet (\lambda_n X)_T \to 0$ in L^0 for every sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathrm{b}\mathcal{E}_0$ with $\|H^n\|_\infty \leqslant 1$. So take such a sequence $(H^n)_{n\in\mathbb{N}}$. Then we have

$$H^n \bullet (\lambda_n X)_T = (\lambda_n H^n) \bullet X_T,$$

and $\|\lambda_n H^n\|_\infty \leqslant |\lambda_n| \to 0$. Since $X \in \mathbb{D}$ is a semimartingale, it is a good integrator (by Theorem 2.7), and thus $(\lambda_n H^n) \bullet X_T \to 0$ in L^0, as required.

Conversely, assume that $d'_E(\lambda_n X, 0) \to 0$ whenever $\lambda_n \to 0$ in $\mathbb{R}$, and suppose for a contradiction that X is not a semimartingale. Then X is not a good integrator (by Theorem 2.5), and so there exists a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathrm{b}\mathcal{E}_0$ with $\|H^n\|_\infty \to 0$, but with $H^n \bullet X_T \not\to 0$ in L^0. Using our assumption with $\lambda_n = \|H_n\|_\infty$ gives $d'_E(\|H_n\|_\infty X, 0) \to 0$. We may assume that $\|H_n\|_\infty > 0$ for all $n \in \mathbb{N}$. Then by applying Lemma 3.2 with the sequence $(\frac{1}{\|H^n\|_\infty} H^n)_{n\in\mathbb{N}} \subseteq \mathrm{b}\mathcal{E}_0$ (which satisfies $\|\frac{1}{\|H^n\|_\infty} H^n\|_\infty = 1$), we conclude that

$$\left(\tfrac{1}{\|H^n\|_\infty} H^n\right) \bullet (\|H^n\|_\infty X)_T \to 0 \quad \text{in } L^0.$$

But $(\frac{1}{\|H^n\|_\infty} H^n) \bullet (\|H^n\|_\infty X)_T = H^n \bullet X_T$, which contradicts $H^n \bullet X_T \not\to 0$ in L^0. This completes the proof.