Coordinator
Benjamin Kotlov

Mathematical Finance

Exercise Sheet 8

Submit by 12:00 on Wednesday, November 22 via the course homepage.

Exercise 8.1 *(Uniqueness of the numéraire portfolio)*

(a) Recall Jensen's inequality: *if X is an integrable random variable taking values in an interval $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ is a convex function such that $f(X)$ is also integrable, then*

$$E[f(X)] \geqslant f(E[X]).$$

Show that if f is strictly convex and X is not almost surely constant (i.e., there exists no $c \in \mathbb{R}$ with $P[X = c] = 1$), we have the strict inequality

$$E[f(X)] > f(E[X]).$$

(b) By using part (a) or otherwise, show that there is at most one numéraire portfolio.

Exercise 8.2 *(Finding the numéraire portfolio)* Show that if Z is an EσMD for S and $1/Z \in \mathcal{X}^1_{++}$, then $1/Z$ is the numéraire portfolio.

Exercise 8.3 *(Yor's formula)* Recall that for a semimartingale X, the *stochastic exponential* of X, denoted by $\mathcal{E}(X)$, is the unique solution Z to the SDE

$$\mathrm{d}Z = Z_- \,\mathrm{d}X, \quad Z_0 = 1.$$

Prove that for two semimartingales X and Y, the following equality holds

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Exercise 8.4 *(Digital option)* In the Black–Scholes model, consider the *digital option* with undiscounted payoff $\widetilde{H} = \mathbf{1}_{\{\widetilde{S}_T > \widetilde{K}\}}$, where $\widetilde{K} > 0$ is fixed. Calculate the arbitrage-free price process and the replicating strategy of the digital option, and thus conclude that it is attainable.