Mathematical Finance

Exercise Sheet 9

Submit by 12:00 on Wednesday, November 29 via the course homepage.

Exercise 9.1 *(Coherent risk measure)* Recall the map $\pi^s : L^\infty \to \mathbb{R}$ defined by

$$\pi^s(H) := \inf\left\{v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H \ P\text{-a.s. for some } \vartheta \in \Theta_{\mathrm{adm}}\right\}.$$

Prove that $\rho := -\pi^s$ is a *coherent risk measure.* That is, for all $H, H' \in L^\infty$,

1. $\pi^s(H) \leqslant \pi^s(H')$ if $H \leqslant H'$ P-a.s. *(monotonicity)*,
2. $\pi^s(H + c) = \pi^s(H) + c$ for all $c \in \mathbb{R}$ *(cash invariance)*,
3. $\pi^s(\lambda H) = \lambda \pi^s(H)$ for all $\lambda > 0$ *(positive homogeneity)*,
4. $\pi^s(H + H') \leqslant \pi^s(H) + \pi^s(H')$ *(subadditivity).*

Deduce that π^s is convex.

What happens in 3. for $\lambda = 0$?

Solution 9.1 Note that $\pi^s(H) \leqslant \|H\|_\infty$ because $\vartheta \equiv 0 \in \Theta_{\mathrm{adm}}$. We check that π^s satisfies the four conditions.

1. Let $v_0 \in \mathbb{R}$ be such that there exists $\vartheta \in \Theta_{\mathrm{adm}}$ with $v_0 + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H'$. Then certainly $v_0 + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H$, and hence $v_0 \geqslant \pi^s(H)$. Taking the infimum over all such $v_0 \in \mathbb{R}$ gives $\pi^s(H') \geqslant \pi^s(H)$ as required.
2. Note that for $v_0 \in \mathbb{R}$ and $\vartheta \in \Theta_{\mathrm{adm}}$, we have
$$v_0 + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H + c \quad \Longleftrightarrow \quad v_0 - c + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H.$$
It follows that the set
$$\left\{v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H + c \ P\text{-a.s. for some } \vartheta \in \Theta_{\mathrm{adm}}\right\}$$
is equal to
$$\left\{v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \,\mathrm{d}S_u \geqslant H \ P\text{-a.s. for some } \vartheta \in \Theta_{\mathrm{adm}}\right\} + c.$$
Taking the infimum over both sets gives $\pi^s(H + c) = \pi^s(H) + c$.

3. Fix $\lambda > 0$ and take $v_0 \in \mathbb{R}$ and $\vartheta \in \Theta_{\text{adm}}$ with $v_0 + \int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant \lambda H$. Then we have $v_0/\lambda + \int_0^T (\vartheta_u/\lambda) \, \mathrm{d}S_u \geqslant H$. Since $\vartheta_u/\lambda \in \Theta_{\text{adm}}$, we have shown that the set
$$\left\{ v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant \lambda H \ P\text{-a.s. for some } \vartheta \in \Theta_{\text{adm}} \right\}$$
is a subset of
$$\lambda \left\{ v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant H \ P\text{-a.s. for some } \vartheta \in \Theta_{\text{adm}} \right\}.$$
We can repeat the above argument to see that the above two sets are indeed equal. Then taking the infimum of both sets gives $\pi^s(\lambda H) = \lambda \pi^s(H)$ as required.

4. Suppose $v_0, v_0' \in \mathbb{R}$ are such that there exist $\vartheta, \vartheta' \in \Theta_{\text{adm}}$ with
$$v_0 + \int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant H \quad \text{and} \quad v_0' + \int_0^T \vartheta_u' \, \mathrm{d}S_u \geqslant H'.$$
Then we have
$$v_0 + v_0' + \int_0^T (\vartheta_u + \vartheta_u') \, \mathrm{d}S_u \geqslant H + H'.$$
As $\vartheta + \vartheta' \in \Theta_{\text{adm}}$, it follows that $v_0 + v_0' \geqslant \pi^s(H + H')$. Taking the infimum over all such v_0 and v_0' gives $\pi^s(H) + \pi^s(H') \geqslant \pi^s(H + H')$ as required.

We have thus shown that $-\pi^s$ is a coherent risk measure. To see that it is convex, take $H, H' \in \Theta_{\text{adm}}$ and $t \in (0,1)$. We have by 4. and 3. that
$$\pi^s\Big(tH + (1-t)H'\Big) \leqslant \pi^s(tH) + \pi^s\Big((1-t)H'\Big) = t\pi^s(H) + (1-t)\pi^s(H'),$$
so that π^s is convex. This completes the proof.

Finally, for $\lambda = 0$, 3. reads $\pi^s(0) = 0$, i.e.
$$\inf \left\{ v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant H \ P\text{-a.s. for some } \vartheta \in \Theta_{\text{adm}} \right\} = 0.$$
First note that $\pi^s(0) \leqslant 0$, as we can take $v_0 = 0$ and $\vartheta \equiv 0$. Now suppose for a contradiction that $\pi^s(0) < 0$. Then there is $v_0 < 0$ with $\int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant -v_0 > 0$ P-a.s. for some $\vartheta \in \Theta_{\text{adm}}$. This violates (NA). So if S satisfies (NA), 3. also holds for $\lambda = 0$.

Exercise 9.2 *(Minimum principle)* Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $X = (X_t)_{t \geqslant 0}$ be a nonnegative RCLL supermartingale. Define the stopping time τ_0 by
$$\tau_0 := \inf\{t \geqslant 0 : X_t \wedge X_{t-} = 0\}.$$

Show that $X \equiv 0$ on $[\![\tau_0, \infty[\![$ P-a.s.

This result is known as the minimum principle for nonnegative supermartingales.

Solution 9.2 Extend X to be a supermartingale on $[0,\infty]$ be setting $X_\infty := 0$. For each $n \in \mathbb{N}$, define the stopping time $\tau_n := \inf\{t \geqslant 0 : X_t < \frac{1}{n}\}$. Note that by right-continuity of X, we have $X_{\tau_n} \leqslant \frac{1}{n}$ on $\{\tau_n < \infty\}$. But also $X_{\tau_n} = 0$ on $\{\tau_n = \infty\}$, and thus $X_{\tau_n} \leqslant \frac{1}{n}$ on all of Ω. Now fix $r \geqslant 0$. As $\tau_n \leqslant \tau_0 \leqslant \tau_0 + r$, we can apply the optional stopping theorem with stopping times $\tau_n \leqslant \tau_0 + r$ to get

$$E[X_{\tau_0+r}] \leqslant E[X_{\tau_n}] \leqslant \frac{1}{n}.$$

Letting $n \to \infty$ gives $E[X_{\tau_0+r}] \leqslant 0$, and as X is nonnegative, this implies that $X_{\tau_0+r} = 0$ P-a.s. Considering the intersection of the events $\{X_{\tau_0+r} = 0\}$ over $r \in \mathbb{Q}^+$ and using right-continuity of X gives the claim.

Exercise 9.3 *(σ-martingales)*

(a) Let $Y = (Y_t)_{0\leqslant t\leqslant T}$ be an RCLL process and $Q \approx P$ an equivalent measure with density process Z given by $Z_t := \frac{dQ}{dP}|_{\mathcal{F}_t}$. Then Y is a Q-σ-martingale if and only if ZY is a P-σ-martingale.

Hint. You may use Bayes theorem and the fact that the sum of two σ-martingales is a σ-martingale.

(b) Show that if S admits a P-equivalent σ-martingale density and $Q \approx P$ on $\mathcal{F}_T$, then S also admits a Q-equivalent σ-martingale density.

Solution 9.3

(a) Suppose first that Y is a Q-σ-martingale. We show that ZY is a P-σ-martingale. Assume for simplicity that $Y_0 = 0$, and write $Y = \psi \bullet M$ for some Q-local martingale M and $\psi \in L(M)$. Applying the stochastic product rule to ZY, we get

$$\mathrm{d}(ZY) = Y_- \,\mathrm{d}Z + Z_- \,\mathrm{d}Y + \mathrm{d}[Z, Y].$$

Note that since $Y = \psi \bullet M$, we have $\mathrm{d}Y = \psi \,\mathrm{d}M$ and hence

$$Z_- \,\mathrm{d}Y = \psi Z_- \,\mathrm{d}M = \psi \,\mathrm{d}(Z_- \bullet M).$$

Also, by again using $Y = \psi \bullet M$, we can write

$$\mathrm{d}[Z, Y] = \mathrm{d}[Z, \psi \bullet M] = \psi \,\mathrm{d}[Z, M].$$

We can thus rewrite $\mathrm{d}(ZY)$ as

$$\mathrm{d}(ZY) = Y_- \,\mathrm{d}Z + \psi \,\mathrm{d}(Z_- \bullet M) + \psi \,\mathrm{d}[Z, M].$$

By applying the stochastic product rule to ZM, we have

$$\mathrm{d}(ZM) = Z_- \,\mathrm{d}M + M_- \,\mathrm{d}Z + \mathrm{d}[M, Z],$$

and hence

$$Z_- \bullet M = ZM - Z_0 M_0 - M_- \bullet Z - [M, Z].$$

We thus have

$$\begin{aligned}\mathrm{d}(ZY) &= Y_- \,\mathrm{d}Z + \psi \,\mathrm{d}\Big(ZM - M_- \bullet Z - [M, Z]\Big) + \psi \,\mathrm{d}[Z, M] \\ &= Y_- \,\mathrm{d}Z + \psi \,\mathrm{d}\Big(ZM - M_- \bullet Z\Big).\end{aligned}$$

Note that as Z is the density process of Q with respect to P, it is a P-martingale. Also, Bayes' theorem implies that ZM is a P-local martingale, since M is a Q-local martingale. Note also that since M_- is locally bounded, the stochastic integral $M_- \bullet Z$ is a P-local martingale. Hence the difference $ZM - M_- \bullet Z$ is a P-local martingale, and thus $\psi \bullet (ZM - M_- \bullet Z)$ is a P-σ-martingale. As $Y_- \bullet Z$ is a P-local martingale, it is a P-σ-martingale, and thus so is ZY, as claimed.

For the converse, simply repeat the above argument, but with Y replaced by ZY and Z replaced by $\frac{1}{Z}$, noting that $\frac{1}{Z}$ is the density process of P with respect to Q, which is a Q-martingale.

(b) We need to show that S admits a Q-equivalent σ-martingale density. Let D denote the given P-equivalent σ-martingale density. Then $D > 0$, D is a P-local martingale and DS is a P-σ-martingale. We define the process $Y := \frac{Z_0}{Z} DS$. Then $ZY = Z_0 DS$ is a P-σ-martingale, so by using part (a), we conclude that Y is a Q-σ-martingale. Also, as D is a P-local martingale, Bayes' theorem implies that $\frac{Z_0}{Z} D$ is a Q-local martingale. Finally, since $\frac{Z_0}{Z}$ is strictly positive (by the minimum principle for nonnegative supermartingales, since $Z_T > 0$) and is 1 at zero, we conclude that $\frac{Z_0}{Z} D$ is an Q-equivalent σ-martingale density for S. This completes the proof.

Exercise 9.4 *(A property of $\mathcal{Z}$)* Fix $Q \in \mathbb{P}_{e,\sigma}(S)$. Recall that for each $t \in [0, T]$, we let $\mathcal{Z}_t$ denote the space of RCLL martingales Z such that $Z_s = \frac{\mathrm{d}R}{\mathrm{d}Q}|_{\mathcal{F}_s}$ for all $0 \leqslant s \leqslant T$ for some $R \in \mathbb{P}_{e,\sigma}(S)$ with $R = Q$ on $\mathcal{F}_t$.

Prove that if $Z^1, Z^2 \in \mathcal{Z}_t$ and $A \in \mathcal{F}_t$, then $Z^1 \mathbf{1}_A + Z^2 \mathbf{1}_{A^c} \in \mathcal{Z}_t$.

Solution 9.4 For notational convenience, we set $Z := Z^1 \mathbf{1}_A + Z^2 \mathbf{1}_{A^c}$. We first show that Z is a martingale. To start, note that since $Z^1_s = Z^2_s = 1$ for all $s \in [0, t]$, then $Z_s = 1$ for $s \in [0, t]$. Since Z^1 and Z^2 are adapted and $A \in \mathcal{F}_t$, it follows that Z is adapted. As Z^1 and Z^2 are RCLL and integrable, then so is Z. It remains to show that Z satisfies the martingale property, i.e. that for all $0 \leqslant s \leqslant u$, we have

$E[Z_u \mid \mathcal{F}_s] = Z_s$. To this end, first note that for $t \leqslant s \leqslant u \leqslant T$, we have $A \in \mathcal{F}_s$ and thus

$$\begin{aligned} E[Z_u|\mathcal{F}_s] &= E[Z_u^1 \mathbf{1}_A + Z_u^2 \mathbf{1}_{A^c}|\mathcal{F}_s] = E[Z_u^1|\mathcal{F}_s]\mathbf{1}_A + E[Z_u^2|\mathcal{F}_s]\mathbf{1}_{A^c} = Z_s^1\mathbf{1}_A + Z_s^2\mathbf{1}_{A^c} \\ &= Z_s. \end{aligned}$$

Next, for $0 \leqslant s \leqslant t \leqslant u$, we use the tower law together with the above to get

$$E[Z_u \mid \mathcal{F}_s] = E\Big[E[Z_u \mid \mathcal{F}_t]\,\Big|\,\mathcal{F}_s\Big] = E[Z_t \mid \mathcal{F}_s] = E[Z_t^1\mathbf{1}_A + Z_t^2\mathbf{1}_{A^c} \mid \mathcal{F}_s] = 1 = Z_s.$$

Lastly, the case $0 \leqslant s \leqslant u \leqslant t$ is trivial, since then $E[Z_u \mid \mathcal{F}_s] = 1 = Z_s$. We have thus shown that Z is an RCLL martingale.

Note also that $Z > 0$ and that $Z \equiv 1$ on $[0, t]$. By Exercise 9.3(a), it suffices to show that ZS is a Q-σ-martingale since we can then conclude that the probability measure R satisfying $\frac{\mathrm{d}R}{\mathrm{d}Q} = Z_T$ is an equivalent σ-martingale measure for S. To this end, first note that since Z^1S and Z^2S are Q-σ-martingales, there exist local martingales M^1, M^2 and positive integrands ψ^1, ψ^2 such that

$$Z^1S - Z_0^1S_0 = \psi^1 \bullet M^1 \quad \text{and} \quad Z^2S - Z_0^2S_0 = \psi^2 \bullet M^2.$$

Using that $Z = Z^1\mathbf{1}_A + Z^2\mathbf{1}_{A^c}$ together with $Z_0^1 = Z_0^2 = Z_0 = 1$, we have

$$\begin{aligned} ZS - Z_0S_0 &= Z^1S\mathbf{1}_A + Z^2S\mathbf{1}_{A^c} - S_0 \\ &= (Z^1S - Z_0^1S_0)\mathbf{1}_A + (Z^2S - Z_0^2S_0)\mathbf{1}_{A^c} \\ &= (\psi^1 \bullet M^1)\mathbf{1}_A + (\psi^2 \bullet M^2)\mathbf{1}_{A^c}. \end{aligned}$$

Now, as $A \in \mathcal{F}_t$, the processes ϕ^1, ϕ^2 defined by

$$\phi^1 := \psi^1\mathbf{1}_{[\![0,t]\!]} + \psi^1\mathbf{1}_{A\times(t,\infty)} \quad \text{and} \quad \phi^2 := \psi^2\mathbf{1}_{A^c\times(t,\infty)}$$

are predictable. By checking the values at times $s \leqslant t$ and $s > t$, we can see that

$$\phi^1 \bullet M^1 + \phi^2 \bullet M^2 = (\psi^1 \bullet M^1)\mathbf{1}_A + (\psi^2 \bullet M^2)\mathbf{1}_{A^c} = ZS - Z_0S_0.$$

Using that $\phi^1 \bullet M^1$ and $\phi^2 \bullet M^2$ are Q-σ-martingales and the fact that the sum of two σ-martingales is a σ-martingale, we conclude that also ZS is a Q-σ-martingale. This completes the proof.