ANALYSIS II - MOCK EXAM - 180 MIN

Overall grade of the exam in the scale 1-6 is computed by rounding $1 + 5P/140$, where P is the number of points obtained (remember that on top of this, your eventual bonus from problem sets will be added).

1. Multiple Choice (MC) — 60 points

Each exercise has some questions that can either be true or false. Each correct answer is worth 1 point. Each unanswered question is worth 0 points. Calling N_{wrong} the number of wrong answers, the corresponding penalization will be computed as follows:

$$\textbf{penalization} = \begin{cases} -0 \textbf{ points} & \textbf{if } 0 \le N_{\text{wrong}} \le 10; \\ -N_{\text{wrong}} + 10 \textbf{ points} & \textbf{if } 11 \le N_{\text{wrong}} \le 20; \\ -2N_{\text{wrong}} + 30 \textbf{ points} & \textbf{if } 21 \le N_{\text{wrong}} \le 30. \end{cases}$$

The number of points in the MC part is computed as the number of correct answers plus the penalization. Nevertheless, the final score of the MC part will never be negative (it will be capped at zero).

Exercise 1. Let $U := \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}$ and $f(x, y) = \sin(xy) - y^4$. Say whether the following statements are true or false.

(1) U is connected. True / False
(2) U is simply connected. True / False
(3) U is compact. True / False
(4) $f(U) \subset \mathbb{R}$ is compact. True / False
(5) $f(U) \subset \mathbb{R}$ is connected. True / False
(6) There are two real numbers $a < b$ such that $f(U) = [a, b]$. True / False

Exercise 2. Let $X \subset \mathbb{R}^2$ with $X \neq \emptyset$ and $X \neq \mathbb{R}^2$.

(1) If X is not open, then it is necessarily closed. True / False
(2) If X is convex, then it is necessarily connected. True / False
(3) If X is bounded, then it is necessarily compact. True / False
(4) If X is complete, then it is necessarily closed. True / False

Exercise 3. You are given some pairs (X, f) where $f : X \to X$ is some continuous function and X is some metric space.

(1) If $X = [0, 1] \subset \mathbb{R}$ and $f(x) := \sin(2x)$, then the Banach Fixed Point Theorem assumptions are satisfied. True / False
(2) If $X = [0, 1] \subset \mathbb{R}$ and $f(x) := x + x^2$, then the Banach Fixed Point Theorem assumptions are satisfied. True / False
(3) If $X = [0, \infty)$ and $f(x) = \frac{x}{100+2x}$, then the Banach Fixed Point Theorem assumptions are satisfied. True / False

Exercise 4. Which of the following formulas are correct for all smooth functions $u, v : \mathbb{R}^n \to (0, \infty)$?

(1) $\partial_i(e^{u^2}) = e^{u^2} 2u\partial_i u$. True / False
(2) $\partial_i \nabla u = \nabla \partial_i u$. True / False
(3) $\partial_i(u/v) = (\partial_i u/v) + (u/\partial_i v)$. True / False
(4) $\operatorname{div}(u\nabla u) = |\nabla u|^2 + uHu$, where Hu is he Hessian matrix of u. True / False

Exercise 5. Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) := \frac{xy^2}{x^2+y^2+z^2}$ for $(x,y,z) \neq (0,0,0)$, and $f(0,0,0) = 0$. Then f is of class

(1) $C^\infty(\mathbb{R}^3 \setminus \{0\})$. True / False
(2) $C^0(\mathbb{R}^3)$. True / False
(3) $C^1(\mathbb{R}^3)$. True / False

Exercise 6. Assume that $f \in C^\infty(\mathbb{R}^3)$ satisfies

$$f(x) = x_1^2 + x_3^4 + o(|x|^5) \qquad \text{as } |x| \to 0,$$

where $x = (x_1, x_2, x_3)$. Then we can deduce that

(1) $\nabla f(0) = (0,0,0)^T$. True / False
(2) $\frac{\partial^2 f}{\partial x_1^2}(0) = 1$. True / False
(3) $\partial_{x_1}\partial_{x_2} f(0) = 0$. True / False
(4) $\lim_{x\to 0} \frac{\partial^2 f}{\partial x_3^2}(x) = 0$. True / False

Exercise 7. Let $f \in C^\infty(\mathbb{R}^3)$ and $\alpha \in \mathbb{R}$ such that

$$\nabla f(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } Hf(0) = \begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where Hf denotes the Hessian matrix. Then

(1) for all $\alpha \in \mathbb{R}$, 0 is not a local maximum point for f. True / False
(2) for all $\alpha > \sqrt{2}$, 0 is a saddle point for f. True / False
(3) for all $\alpha < \sqrt{2}$, 0 is a local minimum point for f. True / False
(4) for $\alpha \neq \pm\sqrt{2}$, the function $\nabla f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism between some open set conaining zero 0 and another open set containing $\nabla f(0)$. True / False

Exercise 8. Consider the set $V := \{(x,y) \in \mathbb{R}^2 : y^4 + y^2 = x^3 + x\}$.

(1) V is a graph with respect to the x-variable around $(1,1)$. True / False
(2) V is a graph with respect to the y-variable around $(0,0)$. True / False
(3) V is a graph with respect to the y-variable around $(1,1)$. True / False
(4) V is a graph with respect to the x-variable around $(0,0)$. True / False

Exercise 9. Consider $F \in C^1(\mathbb{R}^3, \mathbb{R}^2)$ such that

$$F(0,0,1) = (2,-2)^T \text{ and } JF(0,0,1) = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix},$$

and let $M := F^{-1}((2,-2)^T)$. For $t < 1$ let $\gamma(t) := (\sin(2t), t + \frac{1}{2}t^2, \frac{\cos(t)}{1-t})$.

(1) $(F \circ \gamma)'(0) = (0,0)^T$. True / False
(2) In a small neighbourhood of the point $(0,0,1) \in \mathbb{R}^3$, M is a smooth manifold of dimension 2. True / False
(3) In a small neighbourhood of the point $(0,0,1) \in \mathbb{R}^3$, M is a smooth manifold of dimension 1. True / False
(4) The vector $\gamma'(0)$ is normal to M at the point $(0,0,1)$.
(5) There exists $\phi, \psi \in C^1((-\delta,\delta), \mathbb{R})$ such that

$$F(\phi(y), y, \psi(y)) = (2,-2)^T \qquad \text{for all } y \in (-\delta, \delta). \qquad \text{True / False}$$

(6) For sufficiently small $\varepsilon > 0$, the set $F(B_\varepsilon((0,0,1)))$ must be open in $\mathbb{R}^2$. True / False

Exercise 10. Consider the functions

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}, \quad g(x,y) = (x+1)\log(x^2+y^2),$$

and the regions

$$A := \{x^2 + y^2 \geq 1\}, \quad B := \{0 < x^2 + y^2 \leq 1\}.$$

(1) $\int_A |f| < \infty$. True / False
(2) $\int_B |f| < \infty$. True / False
(3) $\int_A |g| = \infty$. True / False
(4) $\int_B |g| = \infty$. True / False

Exercise 11. For $\alpha \in \mathbb{R}$, consider the maps

$$\Phi\colon (x, y) \mapsto (xy, x^2 - y^2) \text{ and } \Psi\colon (x, y) \mapsto (x + y, \alpha x - y, y + 1).$$

(1) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Phi(E)) = 2\int_E (x^2+y^2)\, dxdy$. True / False
(2) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Phi(E)) = 2\int_E (x^2+2y^2)\, dxdy$. True / False
(3) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Psi(E)) = 2\alpha \mathrm{vol}_2(E)$. True / False
(4) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Psi(E)) = \sqrt{2}|\alpha + 1|\mathrm{vol}_2(E)$. True / False
(5) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Psi(E)) = \sqrt{2(\alpha^2 + \alpha + 1)}\mathrm{vol}_2(E)$. True / False

Exercise 12. For $\alpha \geq 0$ consider the integral $I_\alpha := \int_{\mathbb{R}^3} e^{-x^2-y^2-\alpha|z|}\, dx\, dy\, dz$.

(1) $I_0 = \sqrt{\pi}$. True / False
(2) $\alpha I_\alpha = I_1$. True / False
(3) $\lim_{\alpha \to +\infty} (\alpha + \sqrt{\alpha}) I_\alpha = 2\pi$. True / False

Exercise 13. Consider the vector fields

$$X := x_1 e_1, \quad Y := x_2 e_1 \text{ and } Z := \frac{-x_2 e_1 + x_1 e_2}{x_1^2 + x_2^2},$$

all defined in $U := \mathbb{R}^2 \setminus \{0\}$. Let $\gamma \in C^1([0,1], U)$ be any closed curve.

(1) There is necessarily $u \in C^1(U)$ such that $\nabla u = X$ in U. True / False
(2) There is necessarily $v \in C^1(U)$ such that $\nabla v = Y$ in U. True / False
(3) There is necessarily $w \in C^1(U)$ such that $\nabla w = Z$ in U. True / False
(4) Necessarily, $\int_\gamma X \cdot d\gamma = 0$. True / False
(5) Necessarily, $\int_\gamma Y \cdot d\gamma = 0$. True / False
(6) Necessarily, $\int_\gamma Z \cdot d\gamma = 0$. True / False

Exercise 14. Consider the ordinary differential equation

$$y'(x) = F(x, y(x)), \quad y(0) = \alpha.$$

For each of the following choices of F and α, say whether you can use the Cauchy-Lipschitz-Picard-Lindelöf Theorem to deduce, for some small $\delta > 0$, the existence and uniqueness of a solution

$$x \mapsto y(x), \quad x \in I := (-\delta, \delta).$$

(1) $F(x, y) = y^2, \alpha = 0$ True / False
(2) $F(x, y) = x \log y, \alpha = 1$ True / False
(3) $F(x, y) = y + \sqrt{|x|}, \alpha = 0$ True / False
(4) $F(x, y) = x \log y, \alpha = 0$ True / False

2. Box Answer — 20 Points

Only the final answer will be graded in a "All-or-Nothing" fashion.

Exercise 15 (2pt)**.** Let $X := \{(x, y) \mid 1 \leq x \leq 2\} \subset \mathbb{R}^2$. Give an example of a nonconstant $\frac{1}{2}$-Lipschitz contraction $f\colon X \to X$.

Exercise 16 (2pt)**.** Give an explicit example of a continuous and nonconstant function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a compact set $K \subset \mathbb{R}^2$ such that $g^{-1}(K)$ is **not** compact.

Exercise 17 (2pt)**.** Give an explicit example of a C^∞ bijective function $g : \mathbb{R} \to \mathbb{R}$ such that the inverse g^{-1} is not of class C^1.

Exercise 18 (3pt)**.** Give an explicit example of a function $f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ with the following properties: $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact, and f is **not** a diffeomorphism of $\mathbb{R}^2$.

Exercise 19 (2pt)**.** Sketch two open connected sets $U_1, U_2 \subset \mathbb{R}^2$ such that $U_1 \cap U_2$ is not connected and $U_1 \cup U_2$ is connected.

Exercise 20 (3pt)**.** Give an explicit example of a function in $C^2(\mathbb{R}^3) \setminus C^3(\mathbb{R}^3)$.

Exercise 21 (3pt)**.** Let $\phi \colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Give the formula for the normal vector $N \in \mathbb{R}^4$ to the graph of ϕ at a point $X = (x_1, x_2, x_3, \phi(x)) \in \mathbb{R}^4$.

Exercise 22 (3pt)**.** Give an explicit example of a smooth nonconstant curve $\gamma \colon \mathbb{R} \to \mathbb{R}^3$ such that $\gamma(\mathbb{R})$ is **not** a smooth manifold.

3. Short Problems – 40 Points

To earn a full score, you must rigorously prove all your assertions. Each question will be graded separately, so you can assume the results of other questions are given, even if you haven't solved them.

Exercise 23 (11pt)**.** Let $K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $f : K \to \mathbb{R}$ be the function

$$f(x, y) = (x^2 + y^2)e^{\alpha x},$$

where $\alpha \in \mathbb{R}$ is a parameter.

(1) Prove that f attains a maximum and minimum on K. [2pt]
(2) Calculate all the critical points of f in the interior of K, in terms of the parameter α. [4pt]
(3) Calculate the minimum and maximum values of f on K, in terms of the parameter α. [5pt]

Exercise 24 (11pt)**.** Consider the linear differential equation

$$y''(x) - y'(x) = f(x), \quad x \in \mathbb{R}.$$

(1) Determine the most general solution of the associated homogeneous equation. [3pt]
(2) Determine one particular solution in the case $f(x) = e^{-2x}$ [3pt].

Consider the liner system

$$z'(t) = Az(t), \quad A = \begin{pmatrix} \alpha & 1 \\ 0 & -1 \end{pmatrix},$$

(1) Solve the system for $\alpha = -1$ and $z(0) = (1, 1)^T$. [5pt]
(2) Determine the set of α's for which, independently from the intial condition $z(0)$, $|z(t)|$ stays bounded as $t \to +\infty$. [5pt]

Exercise 25 (18pt)**.** Consider the region

$$U := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 < x, 1 < x < 4\},$$

the surface

$$M := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = x, 1 < x < 4\},$$

and the vector fields

$$E(x, y, z) := (x^2 - 5x + 4, -2xy, z)^T, \quad B(x, y, z) := (x^2, -yx, -zx)^T.$$

(1) Sketch the intersection of U with the plane $\{y = 0\}$ and the one with the plane $\{x = 0\}$. Discuss the symmetries of U. Sketch U in 3D perspective with the z axis pointing upwards, the x axis pointing right and the y axis pointing top-right. [4 pt]

(2) Find a vector field A such that

$$B = \operatorname{curl} A,$$

and compute the divergence of B. [4pt]

(Hint: try with $A = (0, -zf(x), yg(x))^T$, for some simple functions $f(x), g(x)$ that you have to find.)

(3) Compute the flow of E across M (the normal of M points outside of U). [4pt] (If needed, you may use the divergence theorem in a piecewise smooth domain without proof.)

(4) Compute the flow of B across M (the normal of M points outside of U). [6pt]

4. Problem — 20 Points

To earn a full score, you must rigorously prove all your assertions. Each question will be graded separately, so you can assume the results of other questions are given, even if you haven't solved them.

Problem 26 (20pt)**.** Define

$$\eta(t) := \begin{cases} \frac{\exp(-\tan^2 t)}{\cos^2 t} & \text{for } t \in (-\pi/2, \pi/2) \\ 0 & \text{for } t \in \mathbb{R} \setminus (-\pi/2, \pi/2). \end{cases}$$

and, for all $\varepsilon > 0$ and $t \in \mathbb{R}$,

$$\eta_\varepsilon(t) := \frac{1}{c_0 \varepsilon} \eta(t/\varepsilon), \qquad c_0 := \int_{\mathbb{R}} \eta(s)\, ds.$$

Let $L = p(d/dt)$ be some linear autonomous differential operator of order $m \geq 1$, with characteristic polynomial $p \in \mathbb{R}[X]$.

Suppose that $u \in C^m(\mathbb{R})$ is a solution of a linear ordinary differential equation $Lu(t) = f(t)$ in the whole $\mathbb{R}$, where $f \in C^0(\mathbb{R})$ is a given function.

Define

$$u_\varepsilon(t) := \int_{-\infty}^{\infty} u(s)\eta_\varepsilon(t-s) ds \text{ and } f_\varepsilon(t) := \int_{-\infty}^{\infty} f(s)\eta_\varepsilon(t-s) ds.$$

(1) Prove that η_ε belongs to $C^1(\mathbb{R})$ and has compact support in $[-\pi\varepsilon/2, \pi\varepsilon/2]$. How would you proof $\eta_\varepsilon \in C^\infty$? [4pt]

(2) Prove that u_ε is of class C^∞ and solves the ODE $Lu_\varepsilon(t) = f_\varepsilon(t)$. [4 pts]

(3) Compute c_0 and prove that $\int_{\mathbb{R}} \eta_\varepsilon(t) dt = 1$ for all $\varepsilon > 0$. [4 pt]

(4) Prove that for any function of polyexponential form $v(t) = \sum q_i(t) \exp(\alpha_i t)$, where $\alpha_i \in \mathbb{C}$ and $q_i \in \mathbb{C}[X]$ the functions $v_\varepsilon(t) := \int_{-\infty}^{\infty} v(s)\eta_\varepsilon(t-s) ds$ are also of polyexponential form.[4 pt]

(5) Prove that, for all $T > 0$,

$$\lim_{k\to\infty} \max_{[-T,T]} |u_\varepsilon - u| \to 0 \text{ as } k \to \infty. \text{ [4 pt]}$$

5. Useful formulas and notation

The following standard notation is used in the whole exam:

- $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$.
- In $\mathbb{R}^2$, a "graph with respect to the x-variable", is any set of the form $\{(x, \phi(x)) : x \in I\}$ for some interval I and some function $\phi \colon I \to \mathbb{R}$.

- $B_r(x)$ denotes the open Euclidean ball of radius $r > 0$ and center $x \in \mathbb{R}^n$.

You can give for granted the following formulas:

- $\int_{\mathbb{R}} e^{-t^2}\, dt = \sqrt{\pi}$.
- If $\Phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $m \geq n$, $K \subset \mathbb{R}^n$ compact, then
$$\mathrm{vol}_n(\Phi(K)) = \int_K \{\det(J\Phi(x)^T \cdot J\Phi(x))\}^{\frac{1}{2}}\, dx.$$
where $J\Phi$ is the Jacobi matrix, where $\partial_{x_i}\Phi^j(x)$ is located in the column i and row j.
- If $A = (A_1, A_2, A_3)^T$ is a vector field in $\mathbb{R}^3$ then
$$\mathrm{curl}\, A = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)^T$$
- $\frac{d}{dt}(\tan t) = \frac{1}{\cos^2 t}$.