# ANALYSIS II - MOCK EXAM - 180 MIN

Overall grade of the exam in the scale 1-6 is computed by rounding $1 + 5P/140$, where $P$ is the number of points obtained (remember that on top of this, your eventual bonus from problem sets will be added).

## 1. Multiple Choice (MC) — 60 points

**Each exercise has some questions that can either be true or false. Each correct answer is worth 1 point. Each unanswered question is worth 0 points. Calling $N_{\text{wrong}}$ the number of wrong answers, the corresponding penalization will be computed as follows:**

$$\textbf{penalization} = \begin{cases} -0 \textbf{ points} & \textbf{if } 0 \le N_{\text{wrong}} \le 10; \\ -N_{\text{wrong}} + 10 \textbf{ points} & \textbf{if } 11 \le N_{\text{wrong}} \le 20; \\ -2N_{\text{wrong}} + 30 \textbf{ points} & \textbf{if } 21 \le N_{\text{wrong}} \le 30. \end{cases}$$

**The number of points in the MC part is computed as the number of correct answers plus the penalization. Nevertheless, the final score of the MC part will never be negative (it will be capped at zero).**

**Exercise 1.** Let $U := \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}$ and $f(x, y) = \sin(xy) - y^4$. Say whether the following statements are true or false.

(1) $U$ is connected. True
(2) $U$ is simply connected. False
(3) $U$ is compact. True
(4) $f(U) \subset \mathbb{R}$ is compact. True
(5) $f(U) \subset \mathbb{R}$ is connected. True
(6) There are two real numbers $a < b$ such that $f(U) = [a, b]$. True

**Exercise 2.** Let $X \subset \mathbb{R}^2$ with $X \neq \emptyset$ and $X \neq \mathbb{R}^2$.

(1) If $X$ is not open, then it is necessarily closed. False
(2) If $X$ is convex, then it is necessarily connected. True
(3) If $X$ is bounded, then it is necessarily compact. False
(4) If $X$ is complete, then it is necessarily closed. True

**Exercise 3.** You are given some pairs $(X, f)$ where $f : X \to X$ is some continuous function and $X$ is some metric space.

(1) If $X = [0, 1] \subset \mathbb{R}$ and $f(x) := \sin(2x)$, then the Banach Fixed Point Theorem assumptions are satisfied. False
(2) If $X = [0, 1] \subset \mathbb{R}$ and $f(x) := x + x^2$, then the Banach Fixed Point Theorem assumptions are satisfied. False
(3) If $X = [0, \infty)$ and $f(x) = \frac{x}{100+2x}$, then the Banach Fixed Point Theorem assumptions are satisfied. True

**Exercise 4.** Which of the following formulas are correct for all smooth functions $u, v : \mathbb{R}^n \to (0, \infty)$?

(1) $\partial_i(e^{u^2}) = e^{u^2} 2u\partial_i u$. True
(2) $\partial_i \nabla u = \nabla \partial_i u$. True
(3) $\partial_i(u/v) = (\partial_i u/v) + (u/\partial_i v)$. False
(4) $\operatorname{div}(u\nabla u) = |\nabla u|^2 + uHu$, where $Hu$ is he Hessian matrix of $u$. False

**Exercise 5.** Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) := \frac{xy^2}{x^2+y^2+z^2}$ for $(x,y,z) \neq (0,0,0)$, and $f(0,0,0) = 0$. Then $f$ is of class

(1) $C^\infty(\mathbb{R}^3 \setminus \{0\})$. True
(2) $C^0(\mathbb{R}^3)$. True
(3) $C^1(\mathbb{R}^3)$. False

**Exercise 6.** Assume that $f \in C^\infty(\mathbb{R}^3)$ satisfies

$$f(x) = x_1^2 + x_3^4 + o(|x|^5) \quad \text{as } |x| \to 0,$$

where $x = (x_1, x_2, x_3)$. Then we can deduce that

(1) $\nabla f(0) = (0,0,0)^T$. True
(2) $\frac{\partial^2 f}{\partial x_1^2}(0) = 1$. False
(3) $\partial_{x_1}\partial_{x_2} f(0) = 0$. True
(4) $\lim_{x\to 0} \frac{\partial^2 f}{\partial x_3^2}(x) = 0$. True

**Exercise 7.** Let $f \in C^\infty(\mathbb{R}^3)$ and $\alpha \in \mathbb{R}$ such that

$$\nabla f(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } Hf(0) = \begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $Hf$ denotes the Hessian matrix. Then

(1) for all $\alpha \in \mathbb{R}$, 0 is not a local maximum point for $f$. True
(2) for all $\alpha > \sqrt{2}$, 0 is a saddle point for $f$. True
(3) for all $\alpha < \sqrt{2}$, 0 is a local minimum point for $f$. False
(4) for $\alpha \neq \pm\sqrt{2}$, the function $\nabla f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism between some open set containing zero 0 and another open set containing $\nabla f(0)$. True

**Exercise 8.** Consider the set $V := \{(x,y) \in \mathbb{R}^2 : y^4 + y^2 = x^3 + x\}$.

(1) $V$ is a graph with respect to the $x$-variable around $(1,1)$. True
(2) $V$ is a graph with respect to the $y$-variable around $(0,0)$. True
(3) $V$ is a graph with respect to the $y$-variable around $(1,1)$. True
(4) $V$ is a graph with respect to the $x$-variable around $(0,0)$. False

**Exercise 9.** Consider $F \in C^1(\mathbb{R}^3, \mathbb{R}^2)$ such that

$$F(0,0,1) = (2,-2)^T \text{ and } JF(0,0,1) = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix},$$

and let $M := F^{-1}((2,-2)^T)$. For $t < 1$ let $\gamma(t) := (\sin(2t), t + \frac{1}{2}t^2, \frac{\cos(t)}{1-t})$.

(1) $(F \circ \gamma)'(0) = (0,0)^T$. False
(2) In a small neighbourhood of the point $(0,0,1) \in \mathbb{R}^3$, $M$ is a smooth manifold of dimension 2. False
(3) In a small neighbourhood of the point $(0,0,1) \in \mathbb{R}^3$, $M$ is a smooth manifold of dimension 1. True
(4) The vector $\gamma'(0)$ is normal to $M$ at the point $(0,0,1)$. False
(5) There exists $\phi, \psi \in C^1((-\delta,\delta), \mathbb{R})$ such that

$$F(\phi(y), y, \psi(y)) = (2,-2)^T \quad \text{for all } y \in (-\delta,\delta). \qquad \text{True}$$

(6) For sufficiently small $\varepsilon > 0$, the set $F(B_\varepsilon((0,0,1)))$ must be open in $\mathbb{R}^2$. True

**Exercise 10.** Consider the functions

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}, \quad g(x,y) = (x+1)\log(x^2+y^2),$$

and the regions

$$A := \{x^2 + y^2 \geq 1\}, \quad B := \{0 < x^2 + y^2 \leq 1\}.$$

(1) $\int_A |f| < \infty$. False
(2) $\int_B |f| < \infty$. True
(3) $\int_A |g| = \infty$. True
(4) $\int_B |g| = \infty$. False

**Exercise 11.** For $\alpha \in \mathbb{R}$, consider the maps

$$\Phi\colon (x,y) \mapsto (xy, x^2 - y^2) \text{ and } \Psi\colon (x,y) \mapsto (x+y, \alpha x - y, y+1).$$

(1) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Phi(E)) = 2\int_E (x^2 + y^2)\, dxdy$. True
(2) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Phi(E)) = 2\int_E (x^2 + 2y^2)\, dxdy$. False
(3) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Psi(E)) = 2\alpha \mathrm{vol}_2(E)$. False
(4) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Psi(E)) = \sqrt{2}|\alpha + 1|\mathrm{vol}_2(E)$. False
(5) If $E \subset \mathbb{R}^2$ is compact then $\mathrm{vol}_2(\Psi(E)) = \sqrt{2(\alpha^2 + \alpha + 1)}\mathrm{vol}_2(E)$. True

**Exercise 12.** For $\alpha \geq 0$ consider the integral $I_\alpha := \int_{\mathbb{R}^3} e^{-x^2 - y^2 - \alpha|z|}\, dx\, dy\, dz$.

(1) $I_0 = \sqrt{\pi}$. False
(2) $\alpha I_\alpha = I_1$. True
(3) $\lim_{\alpha \to +\infty} (\alpha + \sqrt{\alpha}) I_\alpha = 2\pi$. True

**Exercise 13.** Consider the vector fields

$$X := x_1 e_1, \quad Y := x_2 e_1 \text{ and } Z := \frac{-x_2 e_1 + x_1 e_2}{x_1^2 + x_2^2},$$

all defined in $U := \mathbb{R}^2 \setminus \{0\}$. Let $\gamma \in C^1([0,1], U)$ be any closed curve.

(1) There is necessarily $u \in C^1(U)$ such that $\nabla u = X$ in $U$. True
(2) There is necessarily $v \in C^1(U)$ such that $\nabla v = Y$ in $U$. False
(3) There is necessarily $w \in C^1(U)$ such that $\nabla w = Z$ in $U$. False
(4) Necessarily, $\int_\gamma X \cdot d\gamma = 0$. True
(5) Necessarily, $\int_\gamma Y \cdot d\gamma = 0$. False
(6) Necessarily, $\int_\gamma Z \cdot d\gamma = 0$. False

**Exercise 14.** Consider the ordinary differential equation

$$y'(x) = F(x, y(x)), \quad y(0) = \alpha.$$

For each of the following choices of $F$ and $\alpha$, say whether you can use the Cauchy-Lipschitz-Picard-Lindelöf Theorem to deduce, for some small $\delta > 0$, the existence and uniqueness of a solution

$$x \mapsto y(x), \quad x \in I := (-\delta, \delta).$$

(1) $F(x,y) = y^2, \alpha = 0$ True
(2) $F(x,y) = x \log y, \alpha = 1$ True
(3) $F(x,y) = y + \sqrt{|x|}, \alpha = 0$ True
(4) $F(x,y) = x \log y, \alpha = 0$ False

## 2. Box Answer — 20 Points

**Only the final answer will be graded in a "All-or-Nothing" fashion.**

**Exercise 15** (2pt)**.** Let $X := \{(x,y) \mid 1 \leq x \leq 2\} \subset \mathbb{R}^2$. Give an example of a nonconstant $\frac{1}{2}$-Lipschitz contraction $f\colon X \to X$.

*Proof.* $f(x) := (\frac{3}{2}, 0) + 10^{-10} x$. □

**Exercise 16** (2pt)**.** Give an explicit example of a continuous and nonconstant function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a compact set $K \subset \mathbb{R}^2$ such that $g^{-1}(K)$ is **not** compact.

*Proof.* $(x, y) \mapsto (\arctan(x), \arctan(y))$ □

**Exercise 17** (2pt)**.** Give an explicit example of a $C^\infty$ bijective function $g : \mathbb{R} \to \mathbb{R}$ such that the inverse $g^{-1}$ is not of class $C^1$.

*Proof.* $x \mapsto x^5$. □

**Exercise 18** (3pt)**.** Give an explicit example of a function $f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ with the following properties: $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact, and $f$ is **not** a diffeomorphism of $\mathbb{R}^2$.

*Proof.* $(x, y) \mapsto (x + x^2, y)$ (the image is contained in $[-\frac{1}{4}, \infty) \times \mathbb{R}$, and this map sends unbounded sets to unbounded sets). □

**Exercise 19** (2pt)**.** Sketch two open connected sets $U_1, U_2 \subset \mathbb{R}^2$ such that $U_1 \cap U_2$ is not connected and $U_1 \cup U_2$ is connected.

*Proof.* $U_1 := B_1 \setminus \overline{B}_{1/2}$ and $U_2 := (-\frac{1}{10}, \frac{1}{10}) \times \mathbb{R}$. □

**Exercise 20** (3pt)**.** Give an explicit example of a function in $C^2(\mathbb{R}^3) \setminus C^3(\mathbb{R}^3)$.

*Proof.* $(x, y, z) \mapsto \max\{x, 0\}^3$ or $(x, y, z) \mapsto \frac{x^5}{x^2+y^2+z^2}$. □

**Exercise 21** (3pt)**.** Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Give the formula for the normal vector $N \in \mathbb{R}^4$ to the graph of $\phi$ at a point $X = (x_1, x_2, x_3, \phi(x)) \in \mathbb{R}^4$.

*Proof.*

$$N(X) = \frac{1}{\sqrt{1 + \partial_1\phi(x)^2 + \partial_2\phi(x)^2 + \partial_3\phi(x)^2 + \partial_4\phi(x)^2}} \begin{pmatrix} -\partial_1\phi(x) \\ -\partial_2\phi(x) \\ -\partial_3\phi(x) \\ -\partial_4\phi(x) \\ 1 \end{pmatrix}$$

or also

$$N(X) = \frac{1}{\sqrt{1 + |\nabla\phi(x)|^2}} \begin{pmatrix} -\nabla\phi(x) \\ 1 \end{pmatrix}$$

□

**Exercise 22** (3pt)**.** Give an explicit example of a smooth nonconstant curve $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ such that $\gamma(\mathbb{R})$ is **not** a smooth manifold.

*Proof.* $\gamma(t) := (0, 0, t^2)$. □

## 3. Short Problems – 40 Points

**To earn a full score, you must rigorously prove all your assertions. Each question will be graded separately, so you can assume the results of other questions are given, even if you haven't solved them.**

**Exercise 23** (11pt)**.** Let $K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $f : K \to \mathbb{R}$ be the function

$$f(x, y) = (x^2 + y^2)e^{\alpha x},$$

where $\alpha \in \mathbb{R}$ is a parameter.

(1) Prove that $f$ attains a maximum and minimum on $K$. [2pt]
(2) Calculate all the critical points of $f$ in the interior of $K$, in terms of the parameter $\alpha$. [4pt]

(3) Calculate the minimum and maximum values of $f$ on $K$, in terms of the parameter $\alpha$. [5pt]

*Proof.* (1) The set $K$ is compact, so by the Weierstrass Theorem $f$ attains its maximum and minimum value [1pt].

$K$ is closed (being the preimage of the closed interval $(-\infty, 1]$ trough the continuous map $(x, y) \mapsto x^2 + y^2$). $K$ is bounded because by definition its elements have norm less or equal then 1. Hence $K$ is compact [1pt for actually proving that $K$ is compact].

(2) We need to solve the system

$$\begin{cases} 0 = \partial_x f(x,y) = (2x + \alpha x^2 + \alpha y^2)e^{\alpha x}, \\ 0 = \partial_y f(x,y) = 2ye^{\alpha x}, \\ x^2 + y^2 < 1. \end{cases}$$

[2pt for the correct system]

From the second equation, $y = 0$, so we get

$$\begin{cases} 2x + \alpha x^2 = 0, \\ |x| < 1, \end{cases} \Leftrightarrow (x = 0, \alpha \in \mathbb{R}) \text{ or } (x = -\tfrac{2}{\alpha}, |\alpha| > 2).$$

In conclusion the interior critical points are

$$\begin{cases} \{(0,0)\} & \text{if } |\alpha| \le 2, \quad [1pt] \\ \{(0,0), (-\frac{2}{\alpha}, 0)\} & \text{if } |\alpha| > 2. \quad [1pt] \end{cases}$$

(3) The maximum/minimum points of $f$ are either interior critical points or they lie on $\partial K$.

We study $f|_{\partial K}$. Notice that $x^2 + y^2 \equiv 1$ on $\partial K$, so if we set $F(x,y) := e^{\alpha x}$, then

$$\max_{\partial K} f = \max_{\partial K} F \qquad \min_{\partial K} f = \min_{\partial K} f.$$

We study $F$ on $\partial K$ with the method of Lagrange multipliers, the Lagrangian being

$$L(x, y, \lambda) := e^{\alpha x} - \lambda(x^2 + y^2 - 1).$$

We need to solve the system

$$\begin{cases} 0 = \partial_x L(x, y, \lambda) = \alpha e^{\alpha x} - 2\lambda x, \\ 0 = \partial_y L(x, y) = -2\lambda y, \\ 0 = \partial_y L(x, y) = x^2 + y^2 - 1. \end{cases}$$

Since $\lambda \neq 0$ (otherwise the 1st equation would not hold) we find $y = 0$. Then the 3rd equation gives $x = \pm 1$ and the 1st can be solved for $\lambda$. SO the critical points on $\partial K$ must be among

$$(-1, 0) \quad (1, 0).$$

[2pt for correct computation of lagrange multipliers] Since $\partial K$ is compact (it is closed inside a compact set) we known that the extrema of $f|_{\partial K}$ must be attained at such a critical point.

Hence the global extrema of $f$ are to be found among

$$\begin{cases} \{(0,0), (\pm 1, 0)\} & \text{if } |\alpha| \le 2, \\ \{(0,0), (-\frac{2}{\alpha}, 0), (\pm 1, 0)\} & \text{if } |\alpha| > 2. \end{cases}$$

[2pt for correct reasoning among where the max/min can be found] Computing $f$:

$$\begin{cases} \{0, e^{\pm\alpha}\} & \text{if } |\alpha| \le 2, \\ \{0, \frac{4e^{-2}}{\alpha^2}, e^{\pm\alpha}\} & \text{if } |\alpha| > 2. \end{cases}$$

Since $\frac{4e^{-2}}{\alpha^2} < e^{|\alpha|}$ for all $\alpha > 2$, the minimum is always 0 and the maximum always $e^{|\alpha|}$. [1pt for the correct values]

**Alternative for boundary**: we can study $f|_{\partial_K}$ by parametrising the boundary, which is a circle, that is study

$$\max_{\theta\in[0,2\pi]} / \min_{\theta\in[0,2\pi]} f(\cos\theta, \sin\theta) = \max_{\theta\in[0,2\pi]} / \min_{\theta\in[0,2\pi]} e^{\alpha\cos\theta} = e^{|\alpha|}/e^{-|\alpha|}.$$

**Slick solution** [5pt]: notice that for all $(x, y) \in K$ it holds:

$$0 \le f(x, y) = (x^2 + y^2)e^{\alpha x} \le e^{\alpha x} \le e^{|\alpha x|} \le e^{|\alpha|},$$

because $0 \le |x| \le \sqrt{x^2 + y^2} \le 1$. Since $f(0, 0) = 0$ and $f(sign(\alpha), 0) = e^{|\alpha|}$ we deduce that these are the max and the min of $f$ in $K$.

□

**Exercise 24** (11pt)**.** Consider the linear differential equation

$$y''(x) - y'(x) = f(x), \quad x \in \mathbb{R}.$$

(1) Determine the most general solution of the associated homogeneous equation. [3pt]
(2) Determine one particular solution in the case $f(x) = e^{-2x}$ [3pt].

Consider the liner system

$$z'(t) = Az(t), \quad A = \begin{pmatrix} \alpha & 1 \\ 0 & -1 \end{pmatrix},$$

(1) Solve the system for $\alpha = -1$ and $z(0) = (1, 1)^T$. [5pt]
(2) Determine the set of $\alpha$'s for which, independently from the initial condition $z(0)$, $|z(t)|$ stays bounded as $t \to +\infty$. [5pt]

*Proof.* (1) All functions of the form

$$y(x) = Ae^x + B, \qquad A, B \in \mathbb{R}, \text{ [2pt for writing the correct expression]}$$

are solutions. Since they span a 2 dimensional linear space, and the order of the equation (which is linear!) is 2 we deduce that all solutions are of this form. [1 points for motivating why all solutions have that form]

(2) $y(x) := \frac{1}{6}e^{-2x}$ as can be immediately checked by substitution. [3pt, 1pt for the meaningful attempt $\beta e^{-2x}$.]

(3) The system to solve is

$$\begin{cases} z_1' = -z_1' + z_2, \\ z_2' = -z_2, \\ z_1(0) = z_2(0) = 1. \end{cases}$$

The second equation immediately gives $z_2(t) = e^{-t}$. The first becomes

$$z_1' + z_1 = e^{-t}, \quad z_1(0) = 1.$$

This linear ODE with constant coefficients can be solved multiplying both sides by $e^t$ and recognising a derivative on the LHS:

$$(e^t z_1)' = e^t z_1' + e^t z_1 = 1 = (t)' \Rightarrow e^t z_1 = t + C.$$

So one immediately finds from the initial conditions the value of the constant $C = 1$, thus:

$$z_1(t) = te^{-t} + e^{-t}.$$

In other words

$$z(t) = e^{-t}\begin{pmatrix} 1+t \\ 1 \end{pmatrix}.$$

Alternatively one can try to find a particular solution of the form $te^{-t}$.

[3 pt for finding the correct solution, 2 pt for the reasoning. 1pt for writing $te^{-t}$ somewhere].

(4) Let this set be $X \subset \mathbb{R}$. We claim $X = (-\infty, 0]$.

By the theory of linear differential equations, any solution $z(t)$ is of the form

$$z_i(t) = \sum_{j=1}^{2} p_{ij}(t)e^{\lambda_j t}, \quad i = 1, 2.$$

where $\lambda_1, \lambda_2$ are the eigenvalues of $A$, and $\{p_{ij}(t)\}$ are suitable polynomials of degree at most 2. If at least one of the eigenvalue is positive, then for sure $z(t)$ cannot stay bounded (at least for *some* initial conditions the coefficient of the relative exponential will not be the zero polynomial).

Since the spectrum of $A$ is $\{\alpha, 1\}$ this proves that $\alpha \le 0$, i.e., $X \subset (-\infty, 0]$.

We need now to discuss the case $\alpha = 0$. In this case the system is very simple to solve and yields

$$z_2(t) = Ae^{-t}, \quad z_1(t) = B - Ae^{-t},$$

which is always bounded independently from the precise value of the constants $A, B$. [2 pts for trying to work in the poly exp form, 2pt for finding that $\alpha < 0$ works, 1pt for the case $\alpha = 0$]

□

**Exercise 25** (18pt)**.** Consider the region

$$U := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 < x, 1 < x < 4\},$$

the surface

$$M := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = x, 1 < x < 4\},$$

and the vector fields

$$E(x, y, z) := (x^2 - 5x + 4, -2xy, z)^T, \quad B(x, y, z) := (x^2, -yx, -zx)^T.$$

(1) Sketch the intersection of $U$ with the plane $\{y = 0\}$ and the one with the plane $\{x = 0\}$. Discuss the symmetries of $U$. Sketch $U$ in 3D perspective with the $z$ axis pointing upwards, the $x$ axis pointing right and the $y$ axis pointing top-right. [4 pt]

(2) Find a vector field $A$ such that

$$B = \operatorname{curl} A,$$

and compute the divergence of $B$. [4pt]

(Hint: try with $A = (0, -zf(x), yg(x))^T$, for some simple functions $f(x), g(x)$ that you have to find.)

(3) Compute the flow of $E$ across $M$ (the normal of $M$ points outside of $U$). [4pt] (If needed, you may use the divergence theorem in a piecewise smooth domain without proof.)

(4) Compute the flow of $B$ across $M$ (the normal of $M$ points outside of $U$). [6pt]

*Proof.* (1) ... Invariant by rotations in the $(y, z)$ plane.

(2) Computing $\operatorname{curl} A$ we find

$$\operatorname{curl} A = (g(x) + f(x), -zg'(x), -zf'(x)) \text{ [1pt ]}$$

so the choice $f(x) = g(x) = x^2/2$ works and gives

$$A = (0, -zx^2/2, yx^2/2)[2pt].$$

$B$ thus has zero divergence being the curl of something [1pt ].

(3) We use the divergence theorem, since there is no flow of $E$ on the disks bounding the sides of $U$, i.e.: $E_1(1, \cdot, \cdot) = E_1(4, \cdot, \cdot, ) = 0$[2pt]. We compute $\operatorname{div} E = -4$[1pt] and the volume of $U$ is

$$\operatorname{vol}_3(U) = \int_1^4 \Big\{ \iint_{\{z^2+y^2 \le x\}} dy\, dz \Big\}\, dx = \pi \int_1^4 x\, dx = \frac{15\pi}{2},$$

so the flow is $-30\pi$[1pt], by the divergence theorem.

(4) We see our surface as the difference of two embeddings of the two disks:

$$D_1 := \{y^2 + z^2 \le 1\}, \quad D_2 := \{y^2 + z^2 \le 4\},$$

with the map

$$\Phi(y, z) = (y^2 + z^2, y, z).$$

Notice that the normal to $M$ induced by this embedding is the opposite of the outer normal of $U$, that is $\nu_\Phi = -\nu_M$, indeed:

$$(\partial_y \Phi \times \partial_z \Phi)(y, z) = \begin{pmatrix} 2y \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2z \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2y \\ -2z \end{pmatrix}, \quad \nu_\Phi := \frac{\partial_y \Phi \times \partial_z \Phi}{|\partial_y \Phi \times \partial_z \Phi|},$$

which points inside $U$ (draw a picture at $(y, z) = (0, 1)$).

The positively oriented boundaries of $\Phi(D_1), \Phi(D_2)$ are given by

$$\sigma_1(t) := \Phi(\cos t, \sin t) = (1, \cos t, \sin t),$$
$$\sigma_2(t) := \Phi(2\cos t, 2\sin t) = (4, 2\cos t, 2\sin t)$$

with $0 < t < 2\pi$, respectively.

So by Stokes it holds:

$$\begin{aligned}
-\int_M B \cdot \nu_M &= \int_M \operatorname{curl} A \cdot \nu_\Phi = \int_{\Phi(D_2)} \operatorname{curl} A \cdot \nu_\Phi - \int_{\Phi(D_1)} \operatorname{curl} A \cdot \nu_\Phi \\
&= \int_{\Phi(\partial D_2)} A \cdot d\ell - \int_{\Phi(\partial D_1)} A \cdot d\ell \\
&= \int_0^{2\pi} A(\sigma_2(t)) \cdot \sigma_2'(t)\, dt - \int_0^{2\pi} A(\sigma_1(t)) \cdot \sigma_1'(t)\, dt \\
&= \int_0^{2\pi} \begin{pmatrix} 0 \\ -16\sin\theta \\ 16\cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2\sin\theta \\ 2\cos\theta \end{pmatrix} d\theta - \frac{1}{2} \int_0^{2\pi} \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} d\theta \\
&= 32 \int_0^{2\pi} \cos^2\theta + \sin^2\theta\, d\theta - \frac{1}{2} \int_0^{2\pi} \cos^2\theta + \sin^2\theta\, d\theta \\
&= 64\pi - \pi = 63\pi.
\end{aligned}$$

So we find $\int_M B \cdot \nu_M = -63\pi$. [1 point for stokes, 2pt for setting up with difference of domains, 2 pt for correct parametrisation/orientation, 1 point for correct final numbers]

□

## 4. Problem — 20 Points

**To earn a full score, you must rigorously prove all your assertions. Each question will be graded separately, so you can assume the results of other questions are given, even if you haven't solved them.**

**Problem 26** (20pt)**.** Define

$$\eta(t) := \begin{cases} \frac{\exp(-\tan^2 t)}{\cos^2 t} & \text{for } t \in (-\pi/2, \pi/2) \\ 0 & \text{for } t \in \mathbb{R} \setminus (-\pi/2, \pi/2). \end{cases}$$

and, for all $\varepsilon > 0$ and $t \in \mathbb{R}$,

$$\eta_\varepsilon(t) := \frac{1}{c_0 \varepsilon} \eta(t/\varepsilon), \qquad c_0 := \int_{\mathbb{R}} \eta(s)\, ds.$$

Let $L = p(d/dt)$ be some linear autonomous differential operator of order $m \geq 1$, with characteristic polynomial $p \in \mathbb{R}[X]$.

Suppose that $u \in C^m(\mathbb{R})$ is a solution of a linear ordinary differential equation $Lu(t) = f(t)$ in the whole $\mathbb{R}$, where $f \in C^0(\mathbb{R})$ is a given function.

Define

$$u_\varepsilon(t) := \int_{-\infty}^{\infty} u(s)\eta_\varepsilon(t-s)ds \text{ and } f_\varepsilon(t) := \int_{-\infty}^{\infty} f(s)\eta_\varepsilon(t-s)ds.$$

(1) Prove that $\eta_\varepsilon$ belongs to $C^1(\mathbb{R})$ and has compact support in $[-\pi\varepsilon/2, \pi\varepsilon/2]$. How would you prove $\eta_\varepsilon \in C^\infty$? [4pt]
(2) Prove that $u_\varepsilon$ is of class $C^\infty$ and solves the ODE $Lu_\varepsilon(t) = f_\varepsilon(t)$. [4 pts]
(3) Compute $c_0$ and prove that $\int_{\mathbb{R}} \eta_\varepsilon(t)dt = 1$ for all $\varepsilon > 0$. [4 pt]
(4) Prove that for any function of polyexponential form $v(t) = \sum q_i(t) \exp(\alpha_i t)$, where $\alpha_i \in \mathbb{C}$ and $q_i \in \mathbb{C}[X]$ the functions $v_\varepsilon(t) := \int_{-\infty}^{\infty} v(s)\eta_\varepsilon(t-s)ds$ are also of polyexponential form.[4 pt]
(5) Prove that, for all $T > 0$,

$$\lim_{\varepsilon \to 0} \max_{[-T,T]} |u_\varepsilon - u| \to 0 \text{ as } \varepsilon \to 0. \text{ [4 pt]}$$

*Proof.* (1) We first observe that since $\eta(t) = 0$ for all $|t| \geq \pi/2$, then $\eta_\varepsilon(t) =$ 0whenever $|t/\varepsilon| \geq \pi/2$. This exactly means that $\eta$ has compact support in $[-\pi\varepsilon/2, \pi\varepsilon/2]$.

In order to prove that $\eta \in C^0$ we need to prove

$$\lim_{t \to \pm\pi/2} \eta(t) = 0.$$

Indeed:

$$\frac{1}{\cos^2 t} = O(1/t^2), \quad \exp\{-\tan^2 t\} = \exp\{O(1/t^2)\} \text{ as } t \uparrow \pi/2^-,$$

so we find

$$\lim_{t \to \pi/2^-} \frac{\exp(-\tan^2 t)}{\cos^2 t} = 0.$$

Similarly for $t \downarrow \pi/2^+$. Alternatively one can do a substitution:

$$\lim_{t \to \pi/2^-} \frac{\exp(-\tan^2 t)}{\cos^2 t} = \lim_{\cos^2(t) = y \to 0^+} y^{-2} \exp\{\frac{1-y^2}{y^2}\} = 0.$$

In order to prove $C^1$ we need to show that

$$\lim_{t \to \pm\pi/2} \eta'(t) = 0.$$

Computing the derivative

$$\eta'(t) = -2\exp\{-\tan^2 t\}\frac{\sin t\cos t + \tan^3 t}{\cos^4(t)},$$

once again the factor $= \frac{\sin t\cos t+\tan^3 t}{\cos^4(t)}$ has a polynomial divergence at $\pm\pi/2$ while the factor $-2\exp\{-\tan^2 t\}$ has a superexponential damping, thus $\eta'(t) \to 0$.

It is clear that keeping differentiating $\eta$ we will get a more complicated rational function of $\cos t, \sin t$ multiplying the exponential factor $\exp\{-\tan^2 t\}$, which goes to zero faster than any power of $t$, so the same reasoning will apply to all derivatives.

(2) Since $u_\varepsilon$ is an integral over a compact interval depending on a parameter, $u_\varepsilon(t) = \int_{t-\pi}^{t+\pi} F_\varepsilon(s,t)\,ds$, by the theory seen in class it suffices to show that the function

$$F_\varepsilon(s,t) := u(s)\eta_\varepsilon(s-t)$$

is $C^\infty$ in the variable $t$! But, even if $u$ is only of class $C^m$, once we fix $s$, the function $F(s,\cdot)$ has the same regularity of $\eta_\varepsilon$, which is $C^\infty$.

Additionally, $F$ is of class $C^m$ in the joint variables $(s,t)$.

We compute, changing variables ($s = t-r$) and using the differentiation under the integral (which are compactly supported!)

$$\begin{aligned}
Lu_\varepsilon(t) &= \sum_{j=0}^m p_j \frac{d^m}{dt^m}\int_{\mathbb{R}} u(s)\eta_\varepsilon(t-s)\,ds \\
&= \sum_{j=0}^m p_j \frac{d^m}{dt^m}\int_{\mathbb{R}} u(t-r)\eta_\varepsilon(r)\,dr \\
&= \int_{\mathbb{R}} \sum_{j=0}^m p_j \frac{d^m u}{dt^m}(t-r)\eta_\varepsilon(r)\,dr \\
&= \int_{\mathbb{R}} Lu(t-r)\eta_\varepsilon(r)\,dr \\
&= \int_{\mathbb{R}} f(t-r)\eta_\varepsilon(r)\,dr \\
&= \int_{\mathbb{R}} f(s)\eta_\varepsilon(t-r)\,ds = f_\varepsilon(t).
\end{aligned}$$

(3) By the change of variables formula $\tan t = s$, and $dt = \cos^2(t)ds$, we find

$$\begin{aligned}
c_0 = \int_{\mathbb{R}} \eta(t)\,dt &= \int_{-\pi/2}^{\pi/2} \exp\{-\tan^2(t)\}\,\frac{dt}{\cos^2(t)} \\
= \int_{\mathbb{R}} e^{-s^2}\,ds &= \sqrt{\pi}.
\end{aligned}$$

Now, by scaling $t = \varepsilon t'$ we find

$$\int_{\mathbb{R}} \eta_\varepsilon(t)\,dt = \frac{1}{c_0\varepsilon}\int_{\mathbb{R}} \eta(t/\varepsilon)\,dt = \frac{1}{c_0\varepsilon}\int_{\mathbb{R}} \eta(t')\,\varepsilon dt = 1.$$

(4) By the theory of linear differential equations we established that a function is a poly–exponential **if and only if** it solves a linear differential equation with constant coefficients.

Hence, $v$ must solve an appropriate linear ODE $Lv = 0$, and so the previous three points imply that $Lv_\varepsilon = 0$. But then this in turn implies that $v_\varepsilon$ must be of polyexponential form!.

(5) We first realise that

$$u(t) = u(t) \cdot 1 = \int_{\mathbb{R}} u(t)\eta_\varepsilon(r)\, dr.$$

Now using again the shift of variables, estimate for $|t| \leq T$:

$$\begin{aligned} |u_\varepsilon(t) - u(t)| &= \Big| \int_{\mathbb{R}} (u(t-r) - u(t))\eta_\varepsilon(r)\, dr \Big| \\ &\leq \int_{\mathbb{R}} |u(t-r) - u(t)|\eta_\varepsilon(r)\, dr \\ &= \int_{\pi/2}^{\pi/2} |u(t-\varepsilon s) - u(t)|\eta(s)\, ds. \end{aligned}$$

Now we use the mean value inequality

$$|u(t-\varepsilon s) - u(t)| \leq \varepsilon |s| \max_{[t-\pi/2, t+\pi/2]} |u'|$$

so we have

$$\begin{aligned} \sup_{|t|\leq T} |u_\varepsilon(t) - u(t)| &\leq \sup_{|t|\leq T} \int_{\pi/2}^{\pi/2} |u(t-\varepsilon s) - u(t)|\eta(s)\, ds \\ &\leq \varepsilon \big( \sup_{|t|\leq T} \max_{[t-\pi/2, t+\pi/2]} |u'| \big) \int_{\pi/2}^{\pi/2} |s|\eta(s)\, ds \\ &\leq \varepsilon \max_{|\xi|\leq T+\pi} |u'(\xi)| \int_{\pi/2}^{\pi/2} |s|\eta(s)\, ds \leq C\varepsilon \text{ as } \varepsilon \to 0, \end{aligned}$$

for a suitable constant $C$ that depends on $u, T$ and $\eta$.

□

## 5. Useful formulas and notation

The following standard notation is used in the whole exam:

- $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$.
- In $\mathbb{R}^2$, a "graph with respect to the $x$-variable", is any set of the form $\{(x, \phi(x)) : x \in I\}$ for some interval $I$ and some function $\phi \colon I \to \mathbb{R}$.
- $B_r(x)$ denotes the open Euclidean ball of radius $r > 0$ and center $x \in \mathbb{R}^n$.

You can give for granted the following formulas:

- $\int_{\mathbb{R}} e^{-t^2}\, dt = \sqrt{\pi}$.
- If $\Phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $m \geq n$, $K \subset \mathbb{R}^n$ compact, then

$$\mathrm{vol}_n(\Phi(K)) = \int_K \{\det(J\Phi(x)^T \cdot J\Phi(x))\}^{\frac{1}{2}}\, dx.$$

  where $J\Phi$ is the Jacobi matrix, where $\partial_{x_i}\Phi^j(x)$ is located in the column $i$ and row $j$.
- If $A = (A_1, A_2, A_3)^T$ is a vector field in $\mathbb{R}^3$ then

$$\mathrm{curl}\, A = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)^T$$

- $\frac{d}{dt}(\tan t) = \frac{1}{\cos^2 t}$.