Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems marked with (♡).

10.1. BONUS PROBLEM. Calculate the area of the spherical quadrilateral $S \subset \mathbb{S}^2$ bounded by the meridians $-\pi < \phi_1 < \phi_2 < \pi$ and parallels $-\pi/2 < \theta_1 < \theta_2 < \pi/2$.

10.2. The surface area of a Torus. Let $S = \{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 4\}$ and $T = \{x \in \mathbb{R}^3 \,|\, d_S(x) \leq 1\}$, where $d_S(x) = \inf\{|x - y| \,|\, y \in S\}$ denotes the minimal distance from S to $x \in \mathbb{R}^3$. Parameterize ∂T and then calculate the area.

10.3. Solids of revolution. ♡ Given $f \in C^2((a, b)) \cap C([a, b])$ such that $f \geq 0$, define the *rotational body around the x-axis* as

$$\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \mid a < x < b, \sqrt{y^2 + z^2} < f(x)\right\},$$
$$\partial_{\text{side}}\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \mid a < x < b, \sqrt{y^2 + z^2} = f(x)\right\},$$

1. Say whether $\Omega, \partial_{\text{side}}\Omega$ are open/closed/connected subsets of $\mathbb{R}^3$.
2. Show that $\text{vol}_3(\Omega) = \pi \int_a^b f(x)^2 dx$,
3. Show that $\text{vol}_2(\partial_{\text{side}}\Omega) = 2\pi \int_a^b \sqrt{1 + f'(x)^2} f(x) dx$. Hint: parametrise $(x, \theta) \mapsto (x, f(x)\cos\theta, f(x)\sin\theta)$.
4. Calculate the volume and surface area of the 'improper rotational body' that arises when $f(x) = 1/x, a = 1, b = \infty$.
5. (*) Show that Ω is a bounded C^1 domain (as in Definition 14.3) if and only if

$$f(x) > 0 \text{ in } (a, b), \quad f'(a^+) = +\infty, \quad f'(b^-) = -\infty.$$

10.4. Tractrix. Consider the planar curve $\sigma\colon (0, \pi) \to \mathbb{R}^2$ given by $\sigma(t) := (\sin(t), \cos(t) + \log\tan(t/2))$.

1. ♡ Compute the length $L(\sigma|_{[\epsilon,\pi/2]})$ and study its behaviour as $\epsilon \to 0^+$.
2. For any $t \in (0, \pi)$ consider the segment I_t tangent to σ in t and joining $\sigma(t)$ and the y axis. Show that the length of I_t is always 1, independently from t.
3. (*) Say whether the set $\sigma((0, \pi)) \subset \mathbb{R}^2$ is a smooth manifold. Hint: how would it's tangent space be around the point $\sigma(\pi/2)$.

10.5. Logarithmic Spiral. Consider, for $a > 0, b < 0$ the planar curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$\gamma(t) := (ae^{bt}\cos t, ae^{bt}\sin t).$$

1. ♡ Compute $L(\gamma|_{[0,T]})$ and study its behaviour as $T \to \infty$.
2. Express in polar coordinates the set $\gamma(\mathbb{R})$, and sketch it.
3. (*) Say whether $\gamma(\mathbb{R})$ is a smooth submanifold of $\mathbb{R}^2$.

10.6. Moment of inertia of an ellipsoid.

1. Determine the Jacobi determinant of the mapping

$$\Phi\colon \{(s,t) : s > 0,\, 0 < t < 2\pi\} \to \mathbb{R}^2, \quad (s,t) \mapsto (as\cos(t), bs\sin(t))$$

   in terms of the parameters $a, b > 0$.
2. Calculate the polar moment of inertia $J_0 = \int_B x^2 + y^2 \, d\,\mathrm{vol}(x,y)$ of the ellipse $B = \{(x,y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 \leq 1\}$ with semi-axes $a, b > 0$.