Problems marked with a (*) are a bit harder. Hints are available in the next page.

1.1. Recap Questions.

1. Let $\{x_k\}_{k\in\mathbb{N}} \subset [1,2] \subset \mathbb{R}$ be a sequence. Is it necessarily convergent?
2. Describe all differentiable functions $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ such that $f'(x) = 1/x$ for all $x \neq 0$.
3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and bounded, is it true that
$$\int_0^{\pi/2} f(\cos(x))\,dx = \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt?$$
4. Let $f\colon [0,\infty) \to \mathbb{R}$ have $f''(x) > 0$ for all $x \geq 0$. Can f have a global maximum point?
5. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and let $Q := [0,1]\times[0,1]$ be the unit square. How does $\phi(Q)$ look like? What are the possibilities?
6. Consider the vector space V of polynomials of degree less than 4, and the function $D\colon V \to V$ which associates to a polynomial p its derivative p'. Is D a linear map? Is so, what is the dimension of its null space?

1.2. From equations to drawings. Sketch the following subsets of $\mathbb{R}^2, \mathbb{R}^3$.

1. $A := \{(x,y) : x^2+y^2 < 1,\ y \geq x^2\}$
2. $B := \{(x,y) : y^2 = x^3 - x\}$
3. $C := \{(x,y) : \frac{\pi}{6} < \arctan(y/x) < \frac{\pi}{3},\ 0 < x < y\}$
4. $D := \{(x,y,z) : \sqrt{x^2+y^2} = 1 - z, 0 < z < 1\}$
5. $E := \{(x,y,z) : z = \sqrt{1+x^2+y^2}\}$
6. $F := \{(x,y,z) : z = \max\{|x|,|y|\}\}$

1.3. From drawings to equations. Express in terms of functions and equations the following sets.

1. The coordinates of a point in the plane that goes endlessly back and forth from the origin to the point $(-1,1)$.
2. A planar domain with two holes in it.
3. A bi-dimensional disk sitting inside the plane $\{x+y+z=0\}$ in $\mathbb{R}^3$.
4. A Pac-Man shaped domain in the plane.
5. A pyramid, whose base is a square, sitting in $\mathbb{R}^3$.

Hints:

1.2 As ETH students you have a license for Mathematica. All these plots are one-line-commands using that software!

1.2.2 The set is clearly symmetric with respect to $y \mapsto -y$, so it's enough to plot qualitatively the function $y(x) = \sqrt{x^3 - x}$, separately in the intervals $x \in [-1, 0]$ and $x \in [0, \infty)$... From here you can use all the technology from Analysis I...

1.2.4 Try to figure out how D looks layerwise, i.e., as a planar domain once z is kept fixed.

1.2.5 If you known how to plot $z = f(s), s \geq 0$, then the set $\{(x, y, z) : z = f(\sqrt{x^2 + y^2})\}$ is obtained rotating the plot of f around the z-axis...

1.3.1 Construct an appropriate function $f : \mathbb{R} \to \mathbb{R}^2$, where t represents the time and $f(t)$ represents the position of the moving point at time t. You want f to be periodic in t and $f(t)$ to be aligned with the direction $(-1, 0)$

1.3.3 Observation: the intersection of a round ball with a plane is a disk...

1.3.5 Take inspiration from 1.2.4 and 1.2.6...

1. Solutions

Solution of 1.1:

1. No, take for example $\{1 + (-1)^k\}_{k \in \mathbb{N}}$. What is true is that by the Heine-Borel Theorem, $\{x_k\}_{k \in \mathbb{N}}$ has a convergent *subsequence* (cf. Corollary 2.117).

2. f is determined up to a constant in each of the intervals $(-\infty, 0)$ and $(0, \infty)$ (cf. Corollary 5.46). The point is that these two constants may be different, hence the most general f is

$$f(x) = \begin{cases} \alpha + \log x & \text{if } x > 0, \\ \beta + \log(-x) & \text{if } x < 0, \end{cases}$$

   for some $\alpha, \beta \in \mathbb{R}$. Remark: the answer $f(x) = \log |x| + c, c \in \mathbb{R}$ is *wrong.*

3. Yes, this is the change of variables formula in the integrals (cf. Theorem 7.13).

4. Yes, $f := e^{-x}$ has a maximum point at $x = 0$ and $f'' = f > 0$.

5. $\phi(Q)$ is the parallelogram generated by the two vectors $\{\phi(1, 0), \phi(0, 1)\}$. To be precise, if $\phi(1, 0), \phi(0, 1)$ are aligned the parallelogram degenerates in fact into a segment, and if $\phi(1, 0) = \phi(0, 1) = 0$ then it degenerate (even more) into the single point $\{0\}$.

6. Yes, the derivative is a linear operator when acts on functions (cf. Proposition 5.12), and in particular it stays a linear operator when it acts on polynomials. V has dimension 4 since it is generated by the basis $\{1, X, X^2, X^3\}$ (this is ultimately the very definition of polynomial less than 4). One explicitly computes that, trough D,

$$1 \mapsto 0, \quad X \mapsto 1, \quad X^2 \mapsto 2X, \quad X^3 \mapsto 3X^2.$$

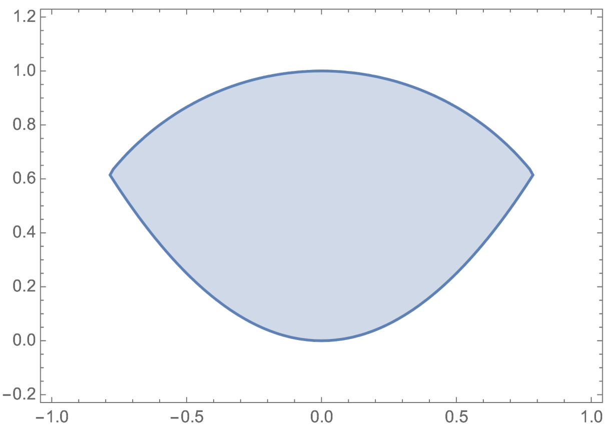
Which reads in matrix form (with respect to the basis $\mathcal{B} := \{1, X, X^2, X^3\}$)

$$[D]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank (=dimension of the span of the columns) of this matrix is 3, so by the rank-nullity Theorem, the nullity is $4 - 3 = 1$.

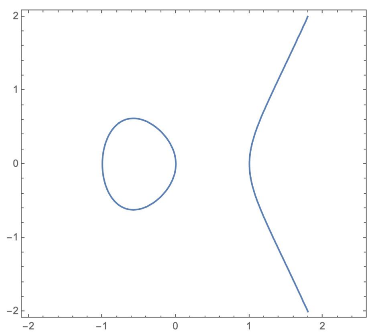
Solution of 1.2:

1. One has to intersect the disk $\{x^2 + y^2 \leq 1\}$ with the epigraph of the parabola $\{y = x^2\}$:



$\{(x, y) : x^2 + y^2 < 1,\ y \geq x^2\} \cap [-1, 1] \times [-.2, 1.2]$

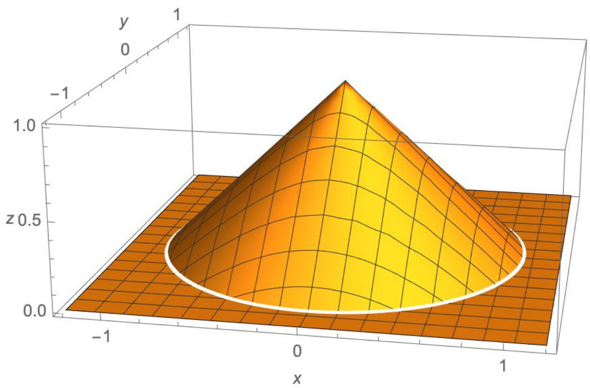
2. Since $x^3 - x = x(x+1)(x-1) \geq 0$ if and only if $x \in [-1, 0] \cup [1, \infty)$, one needs to study the function $f(x) := \sqrt{x^3 - x}$ separately in these two intervals. The set B will then be the union of the graphs of f and $-f$. In the interval $[-1, 0]$, f is concave and vanishes at the extrema, with a maximum point at $x = -\frac{1}{\sqrt{3}}$. Also in $[1, \infty)$ f is concave and monotonically increasing, vanishing at $x = 1$. At infinity it grows like $x^{3/2}$.



$\{y^2 = x^3 - x\} \cap [-2, 2.5] \times [-2, 2]$

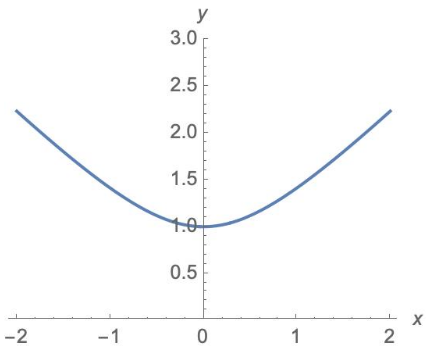
3. Since $0 < x < y$, the set C is contained in the second octant of the plane. The condition $\frac{\pi}{6} < \arctan(y/x) < \frac{\pi}{3}$ forces any point in C to have span an angle between $30°$ and $60°$ with respect to the positive x axis. In summary we find that C is the region of the first quadrant of the plane between the lines $\{y = x\}$ and $\{y = \sqrt{3}x\}$.

4. The set $D \cap \{z = t\}$ is a circle of radius $1 - t$, so the picture must be a round cone of height 1 over the unit circle:

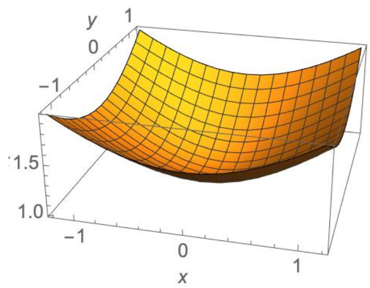


$$\{\sqrt{x^2 + y^2} = 1 - z\} \cap [-1.2, 1.2] \times [-1.2, 1.2] \times [0, 1]$$

5. As in the Hint, the set is obtained rotating the plot of $y = \sqrt{1 + x^2}$ around the z axis:

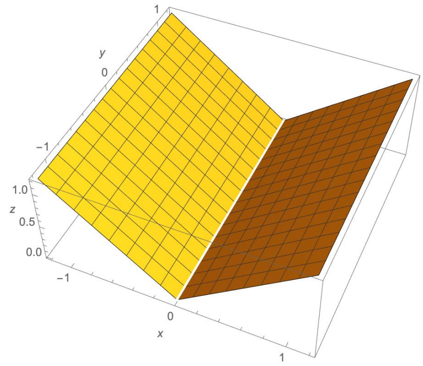


$y = \sqrt{1 + x^2}$

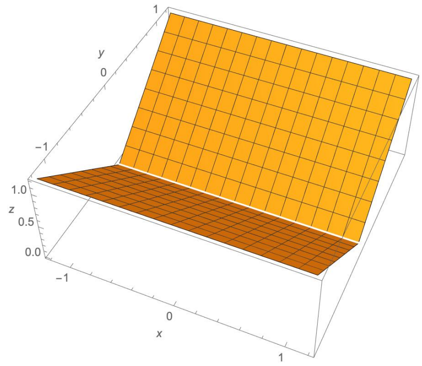


$\{z = \sqrt{1 + x^2 + y^2}\}$

6. First one plots in 3D $z = |x|$ and $z = |y|$, obtaining:

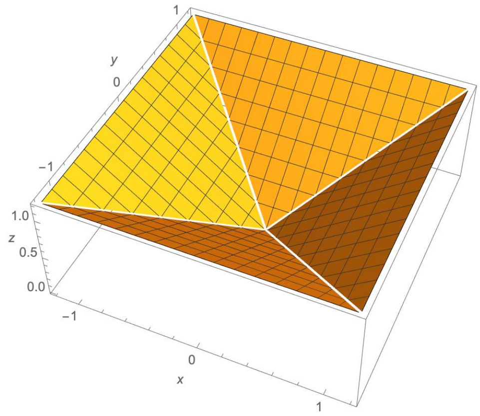


$\{z = |x|\}$



$\{z = |y|\}$

Then one takes the maximum of the two plots. To visualize this, it might be helpful to notice that the level sets $\{\max\{|x|, |y|\} = t\}$ coincide with the squares $[-t, t] \times [-t, t]$, for all $t \geq 0$. One finds



$\{z = \max\{|x|, |y|\}\}$

Solution of 1.3:

1. A point that moves along the line spanned by $(-1, 1)$ is represented by $\mathbb{R} \ni t \mapsto (-f(t), f(t)) \in \mathbb{R}^2$, where $f : \mathbb{R} \to \mathbb{R}$ is some function. We want the motion to take place in the segment, so we require $0 \leq f(t) \leq 1$. We also want the function to be defined for all $t \geq 0$ ("endlessly"). We also want the function to go back and forth, so a natural guess at this point is $f(t) = \sin(t)$... but of course other choices are possible.

2. Remove two small squares from a large square

$$\Big\{10 > \max\{|x|, |y|\} > 1, \max\{|x - 5|, |y - 6|\} > 1\Big\}.$$

3. As suggested it is enough to intersect the ball $\{x^2 + y^2 + z^2 \leq 1\}$ with the plane $\{x + y + z = 0\}$.

4. Remove from a disk a 30° angle:

$$\{x^2 + y^2 \leq 1\} \setminus \{|\arctan(y/x)| \leq \pi/6, x > 0\}.$$

5. We saw that the level sets of $(x, y) \mapsto \max\{|x|, |y|\}$ are squares (cf. 1.2.6), and we saw how to construct a pyramid using a function whose level sets were circles (cf. 1.2.4). Combining the two formulas one gets

$$\Big\{1 - z = \max\{|x|, |y|\}, 0 \leq z \leq 1\Big\}.$$