Some of these problems have a closed-answer format, similar to what you might find on the final exam. "Multiple Choice" means that zero, one or more answers can be correct.

Questions marked with (*) are a bit more complex, you might want to skip them at the first read. Hints available in the next page.

2.1. Examples and Non-examples of Metric spaces. Which of the following pairs are metric spaces? Prove it or provide a counterexample.

1. $(B(X), d)$, where $B(X)$ denotes the set of all bounded functions from a non-empty set X to $\mathbb{R}$ and
$$d(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$
2. $(\mathbb{Q}_+, d)$, where $\mathbb{Q}_+$ are the positive rational numbers and $d(x, y) := \left|\frac{1}{x} - \frac{1}{y}\right|$.
3. $(\mathbb{R}^2, d)$, where $d(x, y) := (x_1 - y_1)^2 + |x_2 - y_2|$.
4. $(\mathbb{R}^2, d)$, where $d(x, y) := |x_1 - y_1|^{1/2} + |x_2 - y_2|$.
5. $(\mathbb{R}^{n\times n}, d)$ with with $d(X, Y) := \Big(\mathrm{Tr}\{(X - Y)^T (X - Y)\}\Big)^{1/2}$.
6. (*) $(\mathbb{R}^{n\times n}, d)$ with
$$d(X, Y) := \sup\{|v^T (X - Y)v| : v \in \mathbb{R}^n, \|v\| = 1\},$$
and $\mathbb{R}^{n\times n}$ denoting the set of square matrices.
7. (*) $(\mathbb{R}^2/\mathbb{Z}^2, d)$ where the flat 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ is the set of equivalence classes of pairs of real numbers under the equivalence relation
$$x, y \in \mathbb{R}^2, x \sim y \Leftrightarrow x_1 - y_1 \in \mathbb{Z}, x_2 - y_2 \in \mathbb{Z},$$
and $d([x], [y]) := \inf_{k,h\in\mathbb{Z}} \|x - y + (k, h)\|$, with $\|\cdot\|$ denoting the Euclidean distance and $[x] \in \mathbb{R}^2/\mathbb{Z}^2$ denoting the equivalence class of $x \in \mathbb{R}^2$.

2.2. Multiple choice. Take a set X and two distances d_1, d_2, so you know that (X, d_1) and (X, d_2) are both metric spaces. Select all the statements below that are necessarily true.

(a) $(X, d_1 + 4d_2)$ is a metric space.

(b) $(X, d_1 \cdot d_2)$ is a metric space.

(c) $(X, \max\{d_1, d_2\})$ is a metric space.

(d) $(X, \min\{d_1, d_2\})$ is a metric space.

2.3. Multiple choice. Let (X, d) be a metric space, and $Y_1, Y_2 \subset X$ subsets. Select all the statements below that are necessarily true.

(a) $\overline{Y_1 \cup Y_2} = \overline{Y_1} \cup \overline{Y_2}$

(b) $\overline{Y_1} \cap \overline{Y_2} \subset \overline{Y_1 \cap Y_2}$

(c) $\overline{Y_1 \cap Y_2} \subset \overline{Y_1} \cap \overline{Y_2}$

(d) $\overline{Y_1 \cap Y_2} = \overline{Y_1} \cap \overline{Y_2}$

2.4. Multiple choice. Let (X, d) be a metric space, and $A \subset X$ a non-empty subset. We define the function "distance from A" as

$$d(\cdot, A) : X \to [0, \infty), \quad d(x, A) := \inf_{a \in A} d(x, a).$$

Select all the statements below that are necessarily true.

(a) If A is closed and $x \in A^c$, then $d(x, A) > 0$.

(b) The set $M := \{x \in X : d(x, A) \geq 1\}$ is closed in X.

(c) For $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$ holds.

(d) If A° is non-empty and $x \in X$, then $d(x, A) = d(x, A^\circ)$.

2.5. Boundary, Interior etc. Determine the interior, closure, and boundary of the following subsets Y of $\mathbb{R}$, for the standard topology on $\mathbb{R}$. No need to justify the answer.

(1) $Y = [0, 1]$
(2) $Y = \mathbb{Q}$
(3) $Y = \emptyset$
(4) $Y = (0, 1)$
(5) $Y = [-1, 1) \setminus \{0\}$
(6) $Y = [0, \infty)$
(7) $Y = \{0\}$
(8) $Y = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\} \right\}$

2.6. Product of metric spaces. Let (X, d_X) and (Y, d_Y) be a pair of metric spaces. Recall that the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is denoted by $X \times Y$. Consider the following functions $X \times Y \to [0, \infty)$:

$$\begin{aligned} d_1((x, y), (x', y')) &:= \max\{d_X(x, x'), d_Y(y, y')\} \\ d_2((x, y), (x', y')) &:= d_X(x, x') + d_Y(y, y') \\ d_3((x, y), (x', y')) &:= \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}. \end{aligned}$$

1. Show that they are all valid distance functions on $X \times Y$.
2. Show that they are all equivalent, i.e., there is a number $C > 0$ such that

$$\begin{aligned} d_1((x, y), (x', y')) &\leq C d_2((x, y), (x', y')) \\ &\leq C^2 d_3((x, y), (x', y')) \leq C^3 d_1((x, y), (x', y')), \end{aligned}$$

   for all $x, x' \in X, y, y' \in Y$.
3. Show that a sequence $(x_n, y_n) \to (x, y)$ with respect to $(X \times Y, d_3)$ if and only if $x_n \to x$ with respect to d_X and $y_n \to y$ with respect to d_Y.

Hints:

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2.1.5 Rewrite for a general matrix $A := X - Y$ what the expression $\mathrm{Tr}\{A^T A\}^{1/2}$ actually means, it should look familiar...

2.1.6 See what happens if $(X - Y)$ is anti-symmetric...

2.1.7 Convince yourself first that the "inf" is in fact a "min"...

2.2.b Compare with 2.1.3...

2.2.d Play with a set of three points (a triangle)...

2.4 First convince yourself with a drawing that the name of this function is appropriate. Then use the characterization of open/closed sets with sequences...

2.6 Don't get distracted by the abstract set-up, you already know all these things for $\mathbb{R} \times \mathbb{R}$, start from there, then re-write those arguments in this general framework.