Some of these problems have a closed-answer format, similar to what you might find on the final exam. "Multiple Choice" means that zero, one or more answers can be correct.

Questions marked with (*) are a bit more complex, you might want to skip them at the first read. Hints available in the next page.

2.1. Examples and Non-examples of Metric spaces. Which of the following pairs are metric spaces? Prove it or provide a counterexample.

1. $(B(X), d)$, where $B(X)$ denotes the set of all bounded functions from a non-empty set X to $\mathbb{R}$ and
$$d(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$
2. $(\mathbb{Q}_+, d)$, where $\mathbb{Q}_+$ are the positive rational numbers and $d(x, y) := \left|\frac{1}{x} - \frac{1}{y}\right|$.
3. $(\mathbb{R}^2, d)$, where $d(x, y) := (x_1 - y_1)^2 + |x_2 - y_2|$.
4. $(\mathbb{R}^2, d)$, where $d(x, y) := |x_1 - y_1|^{1/2} + |x_2 - y_2|$.
5. $(\mathbb{R}^{n\times n}, d)$ with with $d(X, Y) := \left(\mathrm{Tr}\{(X - Y)^T (X - Y)\}\right)^{1/2}$.
6. (*) $(\mathbb{R}^{n\times n}, d)$ with
$$d(X, Y) := \sup\{|v^T (X - Y)v| : v \in \mathbb{R}^n, \|v\| = 1\},$$
and $\mathbb{R}^{n\times n}$ denoting the set of square matrices.
7. (*) $(\mathbb{R}^2/\mathbb{Z}^2, d)$ where the flat 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ is the set of equivalence classes of pairs of real numbers under the equivalence relation
$$x, y \in \mathbb{R}^2, x \sim y \Leftrightarrow x_1 - y_1 \in \mathbb{Z}, x_2 - y_2 \in \mathbb{Z},$$
and $d([x], [y]) := \inf_{k,h\in\mathbb{Z}} \|x - y + (k, h)\|$, with $\|\cdot\|$ denoting the Euclidean distance and $[x] \in \mathbb{R}^2/\mathbb{Z}^2$ denoting the equivalence class of $x \in \mathbb{R}^2$.

2.2. Multiple choice. Take a set X and two distances d_1, d_2, so you know that (X, d_1) and (X, d_2) are both metric spaces. Select all the statements below that are necessarily true.

(a) $(X, d_1 + 4d_2)$ is a metric space.

(b) $(X, d_1 \cdot d_2)$ is a metric space.

(c) $(X, \max\{d_1, d_2\})$ is a metric space.

(d) $(X, \min\{d_1, d_2\})$ is a metric space.

2.3. Multiple choice. Let (X, d) be a metric space, and $Y_1, Y_2 \subset X$ subsets. Select all the statements below that are necessarily true.

(a) $\overline{Y_1 \cup Y_2} = \overline{Y_1} \cup \overline{Y_2}$

(b) $\overline{Y_1} \cap \overline{Y_2} \subset \overline{Y_1 \cap Y_2}$

(c) $\overline{Y_1 \cap Y_2} \subset \overline{Y_1} \cap \overline{Y_2}$

(d) $\overline{Y_1 \cap Y_2} = \overline{Y_1} \cap \overline{Y_2}$

2.4. Multiple choice. Let (X, d) be a metric space, and $A \subset X$ a non-empty subset. We define the function "distance from A" as

$$d(\cdot, A) : X \to [0, \infty), \quad d(x, A) := \inf_{a \in A} d(x, a).$$

Select all the statements below that are necessarily true.

(a) If A is closed and $x \in A^c$, then $d(x, A) > 0$.

(b) The set $M := \{x \in X : d(x, A) \geq 1\}$ is closed in X.

(c) For $x, y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$ holds.

(d) If A° is non-empty and $x \in X$, then $d(x, A) = d(x, A^\circ)$.

2.5. Boundary, Interior etc. Determine the interior, closure, and boundary of the following subsets Y of $\mathbb{R}$, for the standard topology on $\mathbb{R}$. No need to justify the answer.

(1) $Y = [0, 1]$ (2) $Y = \mathbb{Q}$
(3) $Y = \emptyset$ (4) $Y = (0, 1)$
(5) $Y = [-1, 1) \setminus \{0\}$ (6) $Y = [0, \infty)$
(7) $Y = \{0\}$ (8) $Y = \left\{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\right\}$

2.6. Product of metric spaces. Let (X, d_X) and (Y, d_Y) be a pair of metric spaces. Recall that the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is denoted by $X \times Y$. Consider the following functions $X \times Y \to [0, \infty)$:

$$\begin{aligned} d_1((x, y), (x', y')) &:= \max\{d_X(x, x'), d_Y(y, y')\} \\ d_2((x, y), (x', y')) &:= d_X(x, x') + d_Y(y, y') \\ d_3((x, y), (x', y')) &:= \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}. \end{aligned}$$

1. Show that they are all valid distance functions on $X \times Y$.
2. Show that they are all equivalent, i.e., there is a number $C > 0$ such that
$$\begin{aligned} d_1((x, y), (x', y')) &\leq C d_2((x, y), (x', y')) \\ &\leq C^2 d_3((x, y), (x', y')) \leq C^3 d_1((x, y), (x', y')), \end{aligned}$$
for all $x, x' \in X, y, y' \in Y$.
3. Show that a sequence $(x_n, y_n) \to (x, y)$ with respect to $(X \times Y, d_3)$ if and only if $x_n \to x$ with respect to d_X and $y_n \to y$ with respect to d_Y.

Hints:

2.1.3 Ignore what happens in the second variable, is there only to distract you.

2.1.4 Ignore what happens in the second variable, is there only to distract you.

2.1.5 Rewrite for a general matrix $A := X - Y$ what the expression $\mathrm{Tr}\{A^T A\}^{1/2}$ actually means, it should look familiar...

2.1.6 See what happens if $(X - Y)$ is anti-symmetric...

2.1.7 Convince yourself first that the "inf" is in fact a "min"...

2.2.b Compare with 2.1.3...

2.2.d Play with a set of three points (a triangle)...

2.4 First convince yourself with a drawing that the name of this function is appropriate. Then use the characterization of open/closed sets with sequences...

2.6 Don't get distracted by the abstract set-up, you already know all these things for $\mathbb{R} \times \mathbb{R}$, start from there, then re-write those arguments in this general framework.

2. Solutions

Solution of 2.1:

1. This is a metric space, here's the thorough check:

   $\boldsymbol{d<\infty}$**.** For any $f, g \in B(X)$, $d(f,g)$ is the supremum of a set of real numbers (the set of all values $|f(x)-g(x)|$ as x ranges over X), and since the set is bounded (as f and g are bounded functions), $d(f,g)$ is finite.

   Definiteness: By definition, $d(f,g) = \sup_{x\in X} |f(x)-g(x)| \geq 0$, and $d(f,g)=0$ if and only if $|f(x)-g(x)|=0$ for all $x \in X$, which implies $f(x)=g(x)$ for all $x \in X$, i.e., $f=g$.

   Symmetry: For any $f, g \in B(X)$, we have $d(f,g) = \sup_{x\in X} |f(x)-g(x)| = \sup_{x\in X} |g(x)-f(x)| = d(g,f)$.

   Triangle inequality: Let $f, g, h \in B(X)$. Then,

$$\begin{aligned} d(f,h) &= \sup_{x\in X} |f(x)-h(x)| \\ &= \sup_{x\in X} |f(x)-g(x)+g(x)-h(x)| \\ &\leq \sup_{x\in X} (|f(x)-g(x)|+|g(x)-h(x)|) \\ &\leq \sup_{x\in X} |f(x)-g(x)| + \sup_{x\in X} |g(x)-h(x)| = d(f,g)+d(g,h). \end{aligned}$$

2. This is a metric space, here's the thorough check:

   Definiteness: By definition, $d(x,y) = \left|\frac{1}{x}-\frac{1}{y}\right| \geq 0$, and $d(x,y)=0$ if and only if $\frac{1}{x}=\frac{1}{y}$, which implies $x=y$.

   Symmetry: For any $x, y \in \mathbb{Q}_+$, we have $d(x,y) = \left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{y}-\frac{1}{x}\right| = d(y,x)$.

   Triangle inequality: Let $x, y, z \in \mathbb{Q}_+$. Then,

$$\begin{aligned} d(x,z) = \left|\frac{1}{x}-\frac{1}{z}\right| &= \left|\frac{1}{x}-\frac{1}{y}+\frac{1}{y}-\frac{1}{z}\right| \\ &\leq \left|\frac{1}{x}-\frac{1}{y}\right| + \left|\frac{1}{y}-\frac{1}{z}\right| = d(x,y)+d(y,z). \end{aligned}$$

3. This is not a metric space, since the triangular inequality fails, take for example:

$$d((0,0),(1,0)) + d((-1,0),(0,0)) = 2, \quad d((1,0),(-1,0)) = 2^2 = 4.$$

4. This is a metric space. Definiteness and symmetry are immediate to check. Recall that for $a, b \geq 0$ it holds $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, hence for $x, y, z \in \mathbb{R}^2$

$$\begin{aligned} d(x,y) &\leq |x_1 - z_1 + z_1 - y_1|^{1/2} + |x_2 - z_1 + z_1 - y_2| \\ &\leq \Big(|x_1 - z_1| + |z_1 - y_1|\Big)^{1/2} + |x_2 - z_1| + |z_1 - y_2| \\ &\leq |x_1 - z_1|^{1/2} + |z_1 - y_1|^{1/2} + |x_2 - z_1| + |z_1 - y_2| = d(x,z) + d(y,z). \end{aligned}$$

5. This is a metric space, in fact it is $\mathbb{R}^m$ with its standard distance in disguise and $m = n \times n$. Indeed unwinding the definitions:

$$\mathrm{Tr}\{A^T A\} = \sum_{i=1}^n (A^T A)_{ii} = \sum_{i=1}^n (\sum_{k=1}^n A^T_{ik} A_{ki})_{ii} = \sum_{i=1}^n (\sum_{k=1}^n A_{ki} A_{ki})_{ii} = \sum_{i,k=1}^n A^2_{ik}.$$

Thus, if we think X, Y as vectors in $\mathbb{R}^m$ with $m = n \times n$, we find

$$d(X,Y) = \Big(\mathrm{Tr}\{(X-Y)^T (X-Y)\}\Big)^{1/2} = \Big(\sum_{i,k=1}^n (X-Y)^2_{ik}\Big)^{1/2} = \|X - Y\|.$$

6. This is not a metric space, since $d(0, X)$ can be zero for $X \neq 0$, just consider any anti-symmetric[1] matrix X and any $v \in \mathbb{R}^n$:

$$\begin{aligned} |v^T X v| &= \Big|\sum_{i,j=1}^n v_i v_j X_{ij}\Big| = \tfrac{1}{2}\Big|\sum_{i,j=1}^n v_i v_j X_{ij} + \sum_{p,q=1}^n v_p v_q X_{pq}\Big| \\ &= \tfrac{1}{2}\Big|\sum_{i,j=1}^n v_i v_j X_{ij} - \sum_{p,q=1}^n v_q v_p X_{qp}\Big| = \tfrac{1}{2}\Big|\sum_{i,j=1}^n v_i v_j X_{ij} - \sum_{i,j=1}^n v_i v_j X_{ij}\Big| = 0. \end{aligned}$$

Hence[2] $d(X, 0) = \sup_{v \in \mathbb{R}^n} |v^T X v| = 0$.

7. This is a metric space. We first observe that for each $x, y \in \mathbb{R}^2$ there are $(\bar{k}, \bar{h}) in \mathbb{Z}^2$ such that

$$\inf_{(k,h)\in\mathbb{Z}^2} \|x - y - (k,h)\| = \|x - y - (\bar{k}, \bar{h})\|,$$

in other words the "inf" is a "min". To prove this observe that $\inf_{(k,h)\in\mathbb{Z}^2} \|x - y - (k,h)\| \leq \|x - y\| := L$ and that every pair of integers outside the box $Q_L := [-100L, 100L] \times [-100L, 100L]$ will give a much larger result[3]. So the best integers are to be found among the finitely many lying inside Q_L.

[1] That is $X_{ij} = -X_{ji}$ for all $i, j \in \{1, \ldots, n\}$.

[2] In these kind of manipulations, always remember that the name of the index upon which you are summing can be changed without consequences as long as it is changed in all its appearances.

[3] Triangular inequality:

$$\|x - y - (k,h)\| \geq \|(k,h)\| - \|x - y\| \geq 100L - L \geq 10L \text{ for all } (k,h) \in \mathbb{Z}^2 \setminus Q_L.$$

This proves that if $0 = d([x],[y]) = \inf_{(k,h)\in\mathbb{Z}^2} \|x-y-(k,h)\| = 0$, then $x - y = (\bar{k},\bar{h}) \in \mathbb{Z}^2$ which means $[x] = [y]$ (i.e., $x \sim y$). Conversely, if $[x] = [y]$ then by definition there is a pairs of integers $\bar{k},\bar{h}$ such that $x - y = (\bar{k},\bar{h})$ which shows that

$$d([x],[y]) = \inf_{(k,h)\in\mathbb{Z}^2} \|x-y-(k,h)\| \le \|x-y-(\bar{k},\bar{h})\| = 0.$$

We check symmetry:

$$\inf_{(k,h)\in\mathbb{Z}^2} \|x-y-(k,h)\| = \inf_{(k,h)\in\mathbb{Z}^2} \|y-x+(k,h)\| = \inf_{(k',h')\in\mathbb{Z}^2} \|y-x-(k',h')\|.$$

We finally check the triangular inequality. Here we used a slightly more advanced version of the trick of "adding and subtracting the same quantity". For $x,y,z \in \mathbb{R}^2$ and $h,h',k,k' \in \mathbb{Z}$ we write

$$\begin{aligned} d([x],[y]) &\le \|x-y-(k+k',h+h')\| = \|x-z+z-y-(k,h)+(k',h')\| \\ &\le \|x-z-(k,h)\| + \|z-y-(k',h')\|. \end{aligned}$$

Now the choice of k,k',h,h' is completely free, so we can choose (k,h) that realize $d([x],[z])$ and (k',h') that realize $d([z],[y])$.

Solution of 2.2:

(a) True. Definiteness and symmetry are simple. Given $x,y,z \in X$ one has

$$\begin{aligned} d_1(x,y) &\le d_1(x,z) + d_1(z,y) \\ d_2(x,y) &\le d_2(x,z) + d_2(z,y) \end{aligned}$$

thus summing them

$$d_1(x,y) + 4d_2(x,y) \le d_1(x,z) + d_1(z,y) + 4d_2(x,z) + 4d_2(z,y).$$

(b) False. Counterexample: $X = \mathbb{R}, d_1 = d_2 = |x-y|$. Compare with 2.1.3.

(c) True. Definiteness and symmetry are simple. Given $x,y,z \in X$ one has

$$\begin{aligned} d_1(x,y) &\le d_1(x,z) + d_1(z,y) \le \max\{d_1(x,z),d_2(x,z)\} + \max\{d_1(z,y),d_2(z,y)\}, \\ d_2(x,y) &\le d_2(x,z) + d_2(z,y) \le \max\{d_1(x,z),d_2(x,z)\} + \max\{d_1(z,y),d_2(z,y)\}. \end{aligned}$$

Thus the same upper bound holds for $\max\{d_1(x,y),d_2(x,y)\}$.

(d) False. Take $X = \{A,B,C\}$ and the distances

$$d_1(A,B) := 1, \quad d_1(B,C) := 1, \quad d_1(C,A) := \tfrac{1}{100}$$

and

$$d_2(A,B) := 1, \quad d_2(B,C) := \tfrac{1}{2}, \quad d_2(C,A) := \tfrac{1}{2}.$$

It is clear that $\{1, \frac{1}{2}, \frac{1}{100}\}$ cannot be the set of lengths of the sides of a triangle.

Solution of 2.3:

(a) True. If x is an accumulation point of $Y_1 \cup Y_2$ it means that it is an accumulation point at least for one of them.

(b) False. In $\mathbb{R}$ with the standard topology take $Y_1 := \mathbb{Q}$ and $Y_2 := \mathbb{R} \setminus \mathbb{Q}$. These two sets have empty intersection, but they are both dense. So the right hand side is empty but the left hand side is $\mathbb{R}$.

(c) True. If x is an accumulation point of elements that belong to both Y_1 and Y_2, than it is, in particular, an accumulation point of elements of Y_1 and of elements of Y_2.

(d) False, because it would imply that (b) holds, and we have given a counterexample to it.

Solution of 2.4:

(a) True. If we had $d(x, A) = 0$ then there would be a sequence $\{a_k\} \subset A$ such that $d(x, a_k) \to 0$. This means that $a_k \to x$. On the other hand, since A is closed, this would imply $x \in A$ (cf. Lemma 9.46), which contradicts $x \in A^c$.

(b) True. Notice that $d(x, A) \geq 1$ if and only if $d(x, a) \geq 1$ for all $a \in A$. This means that

$$M := \bigcap_{a \in A} \{x \in X : d(x, a) \geq 1\} = \bigcap_{a \in A} f_a^{-1}([1, \infty)),$$

where $f_a \colon x \mapsto d(x, a)$. But each of the f_a is continuous and $[1, \infty)$ is closed, so M is an intersection of closed sets, hence M is closed (cf. Proposition 9.41).

(c) True. Take any $a \in A$ and write the triangular inequality

$$d(x, a) \leq d(x, y) + d(y, a),$$

and we conclude taking the infimum over $a \in A$.

(d) False. Take $\mathbb{R}$ with the standard distance and $A := \mathbb{Q} \cup [2, 3]$ so that $A^\circ = (2, 3)$. Then $d(0, A) = 0$ but $d(0, (2, 3)) = 2$.

Solution of 2.5:

(1)	$Y^\circ = (0,1)$,	$\overline{Y} = [0,1]$,	$\partial Y = \{0,1\}$,
(2)	$Y^\circ = \emptyset$,	$\overline{Y} = \mathbb{R}$,	$\partial Y = \mathbb{R}$,
(3)	$Y^\circ = \emptyset$,	$\overline{Y} = \emptyset$,	$\partial Y = \emptyset$,
(4)	$Y^\circ = (0,1)$,	$\overline{Y} = [0,1]$,	$\partial Y = \{0,1\}$,
(5)	$Y^\circ = (-1,0) \cup (0,1)$,	$\overline{Y} = [-1,1]$,	$\partial Y = \{-1,0,1\}$,
(6)	$Y^\circ = (0,\infty)$,	$\overline{Y} = [0,\infty)$,	$\partial Y = \{0\}$,
(7)	$Y^\circ = \emptyset$,	$\overline{Y} = \{0\}$,	$\partial Y = \{0\}$,
(8)	$Y^\circ = \emptyset$,	$\overline{Y} = Y \cup \{0\}$,	$\partial Y = \overline{Y}$.

Solution of 2.6:

1. Symmetry and Definiteness are simple to check, so we focus on the triangular inequality. Before embarking in long strings of symbols let us inspect the three functions that we are using

$$\max\{a,b\}, \quad a+b, \quad \sqrt{a^2+b^2}, \quad \text{for } a,b\geq 0.$$

Let us call $F(a,b)$ any of these functions, so that

$$d_i((x,y),(x',y')) = F(d_X(x,x'),d_Y(y,y')).$$

First, F is increasing separately in each of its arguments. Second F has a subadditive property:

$$F(a+a',b+b') \leq F(a,b)+F(a',b'),$$

that is we have:

$$\begin{aligned}\max\{a+a',b+b'\} &\leq \max\{a,b\}+\max\{a',b'\},\\ (a+a')+(b+b') &\leq (a+b)+(a'+b'),\\ \sqrt{(a+a')^2+(b+b')^2} &\leq \sqrt{a^2+b^2}+\sqrt{a'^2+b'^2}.\end{aligned}$$

Checking these inequalities is the core of this exercise. The second is obvious and the third is the triangular inequality in $\mathbb{R}^2$, which has been proven in class (cf., Proposition 9.2). We prove the first: summing the two inequalities

$$a\leq \max\{a,b\}, \quad a'\leq \max\{a',b'\},$$

we find $a+a'\leq \max\{a,b\}+\max\{a',b'\}$. Summing the two inequalities

$$b\leq \max\{a,b\}, \quad b'\leq \max\{a',b'\},$$

we find $b+b'\leq \max\{a,b\}+\max\{a',b'\}$, and so

$$\max\{a+a',b+b'\}\leq \max\{a,b\}+\max\{a',b'\}.$$

Now, we are ready to prove the triangular inequality in the product space. If we write the triangular inequality for X and Y we have

$$\begin{aligned} d_X(x,x'') &\leq \underbrace{d_X(x,x')}_{=:a}+\underbrace{d_X(x',x'')}_{=:a'} = a+a',\\ d_Y(y,y'') &\leq \underbrace{d_Y(y,y')}_{=:b}+\underbrace{d_Y(y',y'')}_{=:b'} = b+b'.\end{aligned}$$

So we can write

$$\begin{aligned} F(d_X(x,x''),d_Y(y,y'')) &\leq F(d_X(x,x')+d_X(x',x''),d_Y(y,y')+d_Y(y',y''))\\ &= F(a+a',b+b')\\ &\leq F(a,b)+F(a',b')\\ &= F(d_X(x,x'),d_Y(y,y'))+F(d_X(x'',x'),d_Y(y'',y')).\end{aligned}$$

2. We just observe that for all $a, b \geq 0$ it holds

$$\max\{a,b\} \leq a+b \leq 2\sqrt{a^2+b^2} \leq 2\sqrt{2\max\{a^2,b^2\}} = 2\sqrt{2}\max\{a,b\}. \tag{1}$$

3. If $x_n \to x$ and $y_n \to y$ it means that

$$a_n := d_X(x_n, x) \to 0 \text{ and } b_n := d_Y(y_n, y) \to 0.$$

Thus by (1), we find that

$$\max\{a_n, b_n\} \to 0, \quad a_n + b_n \to 0 \text{ and } \sqrt{a_n^2 + b_n^2} \to 0,$$

which means that $(x_n, y_n) \to (x, y)$ in $X \times Y$ with respect to all the distances d_i.

Conversely, if at least one of the sequences

$$\max\{a_n, b_n\}, \quad a_n + b_n, \text{ and } \quad \sqrt{a_n^2 + b_n^2},$$

is infinitesimal, one necessarily has that also $\{a_n\}$ and $\{b_n\}$ are infinitesimal.