Some of these problems have a closed-answer format, similar to what you might find on the final exam.

Questions marked with (*) are a bit more complex, you might want to skip them at the first read. Hints available in the next page.

3.1. BONUS PROBLEM. Give an example of:

(a) A $\frac{1}{2}$-Lipschitz function $f\colon [0,1) \to [0,1)$ that has no fixed point.

(b) A map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that has no fixed point, but it is an isometry, i.e.,

$$\|f(x) - f(x')\| = \|x - x'\| \text{ for all } x, x' \in \mathbb{R}^2.$$

3.2. Continuity of the distance. Let (X, d) be a metric space, and endow $X \times X$ with the product distance $d_2(x, y) := d(x_1, y_1) + d(x_2, y_2)$. Show that the distance function $d\colon X \times X \to \mathbb{R}$ is continuous with respect to d_2 and, more precisely, that it is 1-Lipschitz. Is d also continuous with respect to the distance

$$d_1(x, y) := \max\{d(x_1, y_1), d(x_2, y_2)\}?$$

Is d also Lipschitz continuous with respect to the distance

$$d_3(x, y) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}?$$

(We are using a notation consistent with Problem 2.6).

3.3. Continuity of the composition. Let X, Y, Z be metric spaces and let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous functions. Show that $g \circ f\colon X \to Z$ is continuous using at least two of the three equivalent definitions of continuity seen in class.

3.4. Problems at the origin. Show that there is no continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = xy/(x^2 + y^2)$ for all $(x, y) \neq (0, 0)$. On the other hand, show that there is exactly one continuous function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y) = xy/\sqrt{x^2 + y^2}$ for all $(x, y) \neq (0, 0)$.

3.5. Heine-Cantor. Let X, Y be metric spaces and let $f\colon X \to Y$ be a continuous. Show that if X is compact then f is uniformly continuous.

3.6. Open, closed, complete and compact. For each of the following subsets of $\mathbb{R}^N$ say whether they are open/closed/none/compact (with respect to the standard Euclidean structure). Try to prove "efficiently" your assertions.

1. $E_1 := \{x \in \mathbb{R}^3 : 0 < x_2 \leq 2\}$
2. $E_2 := \{x \in \mathbb{R}^n : \sin(\|x\|) \geq \frac{1}{4}\}$

3. $E_3 := \bigcup_{n\geq 1}\{x \in \mathbb{R}^3 : x_1^4 - \frac{1}{n}x_3^2 - x_2^2 > \frac{1}{n}, x_1 + x_2 < 6\}$
4. $E_4 := \{(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) : x \cdot y > \frac{1}{2}\|x\|\|y\|\}$
5. $E_5 := \{X \in \mathbb{R}^{n\times n} :$ the matrix X is invertible$\}$
6. $E_6 := \{X \in \mathbb{R}^{n\times n} :$ the matrix X is symmetric$\}$
7. $E_7 := \{X \in \mathbb{R}^{n\times n} :$ the entries of the matrix X are either 0 or 1$\}$
8. $E_8 := \{x \in \mathbb{R}^n : |x_i| \leq 6,$ for all $1 \leq i \leq n\}$
9. $E_9 := E_8 \times E_3 \subset \mathbb{R}^{n+3}$
10. $E_{10} := E_2 \cap E_8 \subset \mathbb{R}^n$

3.7. Multiple choice. Select all the statements below that are true.

(a) A nonempty open strict subset of $\mathbb{R}^n$ cannot be compact.

(b) An nonempty open strict subset of $\mathbb{R}^n$ cannot be complete.

(c) A complete subset of $\mathbb{R}^n$ contains all its accumulation points.

(d) Countable union of complete subsets of $\mathbb{R}^n$ is complete.

(f) A subset is closed in $\mathbb{R}^n$ if and only if it is complete as metric space itself (with the distance inherited from $\mathbb{R}^n$).

3.8. Multiple choice. Select all the statements below that are true.

(a) If you spread a map of Zurich on your desk, then one point of the desk will coincide with its representation on the map (in an ideal world).

(b) If $f\colon [0, 1] \to [0, 2]$ is $\frac{1}{2}$-Lipschitz, then f has a fixed point.

(c) If $f\colon [0, 2] \to [0, 1]$ is $\frac{1}{2}$-Lipschitz, then f has a fixed point.

(d) If $f\colon [0, 1] \cup [2, 3] \to [0, 1] \cup [2, 3]$ is continuously differentiable with $|f'(x)| < 1$ for all $x \in [0, 1] \cup [2, 3]$, then it has a fixed point.

(e) (*) If $f\colon [0, 1] \to [0, 1]$ is differentiable with $|f'(x)| < 1$ for all $x \in (0, 1)$, then it has a fixed point.

3.9. Multiple choice. Select all the statements below that are true.

(a) The function $(x, y) \mapsto x + y$ is uniformly continuous in $\mathbb{R}^2$.

(b) The function $(x, y) \mapsto xy$ is uniformly continuous in $\mathbb{R}^2$.

(c) The function $(x, y) \mapsto x + y$ is uniformly continuous in $[0, 1]^2$.

(d) The function $(x, y) \mapsto xy$ is uniformly continuous in $[0, 1]^2$.

Hints:

3.2 Compare with Problem 2.6...

3.4 If a function is continuous at 0 and $x_k \to 0$ and $y_k \to$, then $f(x_k)$ and $f(y_k)$ must be converging to the same value...

3.5 For any $\epsilon > 0$ and any $x \in X$ there is some $\delta_x > 0$ such that $f(B(x, \delta_x) \subset B(f(x), \epsilon)$... Now look at the collection $\{B(x, \delta_x)\}_{x \in X}$... Which definition of compactness seems the most useful?