Some of these problems have a closed-answer format, similar to what you might find on the final exam.

Questions marked with (*) are a bit more complex, you might want to skip them at the first read. Hints available in the next page.

3.1. BONUS PROBLEM. Give an example of:

(a) A $\frac{1}{2}$-Lipschitz function $f\colon [0,1) \to [0,1)$ that has no fixed point.

(b) A map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that has no fixed point, but it is an isometry, i.e.,

$$\|f(x) - f(x')\| = \|x - x'\| \text{ for all } x, x' \in \mathbb{R}^2.$$

3.2. Continuity of the distance. Let (X, d) be a metric space, and endow $X \times X$ with the product distance $d_2(x, y) := d(x_1, y_1) + d(x_2, y_2)$. Show that the distance function $d\colon X \times X \to \mathbb{R}$ is continuous with respect to d_2 and, more precisely, that it is 1-Lipschitz. Is d also continuous with respect to the distance

$$d_1(x, y) := \max\{d(x_1, y_1), d(x_2, y_2)\}?$$

Is d also Lipschitz continuous with respect to the distance

$$d_3(x, y) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}?$$

(We are using a notation consistent with Problem 2.6).

3.3. Continuity of the composition. Let X, Y, Z be metric spaces and let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous functions. Show that $g \circ f\colon X \to Z$ is continuous using at least two of the three equivalent definitions of continuity seen in class.

3.4. Problems at the origin. Show that there is no continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = xy/(x^2 + y^2)$ for all $(x, y) \neq (0, 0)$. On the other hand, show that there is exactly one continuous function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y) = xy/\sqrt{x^2 + y^2}$ for all $(x, y) \neq (0, 0)$.

3.5. Heine-Cantor. Let X, Y be metric spaces and let $f\colon X \to Y$ be a continuous. Show that if X is compact then f is uniformly continuous.

3.6. Open, closed, complete and compact. For each of the following subsets of $\mathbb{R}^N$ say whether they are open/closed/none/compact (with respect to the standard Euclidean structure). Try to prove "efficiently" your assertions.

1. $E_1 := \{x \in \mathbb{R}^3 : 0 < x_2 \leq 2\}$
2. $E_2 := \{x \in \mathbb{R}^n : \sin(\|x\|) \geq \frac{1}{4}\}$

3. $E_3 := \bigcup_{n\geq 1}\{x \in \mathbb{R}^3 : x_1^4 - \frac{1}{n}x_3^2 - x_2^2 > \frac{1}{n}, x_1 + x_2 < 6\}$
4. $E_4 := \{(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) : x \cdot y > \frac{1}{2}\|x\|\|y\|\}$
5. $E_5 := \{X \in \mathbb{R}^{n\times n} :$ the matrix X is invertible$\}$
6. $E_6 := \{X \in \mathbb{R}^{n\times n} :$ the matrix X is symmetric$\}$
7. $E_7 := \{X \in \mathbb{R}^{n\times n} :$ the entries of the matrix X are either 0 or 1$\}$
8. $E_8 := \{x \in \mathbb{R}^n : |x_i| \leq 6,$ for all $1 \leq i \leq n\}$
9. $E_9 := E_8 \times E_3 \subset \mathbb{R}^{n+3}$
10. $E_{10} := E_2 \cap E_8 \subset \mathbb{R}^n$

3.7. Multiple choice. Select all the statements below that are true.

(a) A nonempty open strict subset of $\mathbb{R}^n$ cannot be compact.

(b) An nonempty open strict subset of $\mathbb{R}^n$ cannot be complete.

(c) A complete subset of $\mathbb{R}^n$ contains all its accumulation points.

(d) Countable union of complete subsets of $\mathbb{R}^n$ is complete.

(f) A subset is closed in $\mathbb{R}^n$ if and only if it is complete as metric space itself (with the distance inherited from $\mathbb{R}^n$).

3.8. Multiple choice. Select all the statements below that are true.

(a) If you spread a map of Zurich on your desk, then one point of the desk will coincide with its representation on the map (in an ideal world).

(b) If $f\colon [0,1] \to [0,2]$ is $\frac{1}{2}$-Lipschitz, then f has a fixed point.

(c) If $f\colon [0,2] \to [0,1]$ is $\frac{1}{2}$-Lipschitz, then f has a fixed point.

(d) If $f\colon [0,1] \cup [2,3] \to [0,1] \cup [2,3]$ is continuously differentiable with $|f'(x)| < 1$ for all $x \in [0,1] \cup [2,3]$, then it has a fixed point.

(e) (*) If $f\colon [0,1] \to [0,1]$ is differentiable with $|f'(x)| < 1$ for all $x \in (0,1)$, then it has a fixed point.

3.9. Multiple choice. Select all the statements below that are true.

(a) The function $(x,y) \mapsto x + y$ is uniformly continuous in $\mathbb{R}^2$.

(b) The function $(x,y) \mapsto xy$ is uniformly continuous in $\mathbb{R}^2$.

(c) The function $(x,y) \mapsto x + y$ is uniformly continuous in $[0,1]^2$.

(d) The function $(x,y) \mapsto xy$ is uniformly continuous in $[0,1]^2$.

Hints:

3.2 Compare with Problem 2.6...

3.4 If a function is continuous at 0 and $x_k \to 0$ and $y_k \to$, then $f(x_k)$ and $f(y_k)$ must be converging to the same value...

3.5 For any $\epsilon > 0$ and any $x \in X$ there is some $\delta_x > 0$ such that $f(B(x, \delta_x) \subset B(f(x), \epsilon)$... Now look at the collection $\{B(x, \delta_x)\}_{x \in X}$... Which definition of compactness seems the most useful?

3. Solutions

Solution of 3.1: (a) $f(x) := \frac{1+x}{2}$. (b) $f(x,y) := (x+1,y)$.

Solution of 3.2: Recall that the triangular inequality for d entails

$$|d(x,y) - d(x',y)| \leq d(x,x') \text{ for all } x,x',y \in X.$$

So we have for all $x,x',y,y' \in X$ that

$$\begin{aligned} |d(x,x') - d(y,y')| &= |d(x,x') - d(x,y') + d(x,y') - d(y,y')| \\ &\leq |d(x,x') - d(x,y')| + |d(x,y') - d(y,y')| \\ &\leq d(x',y') + d(x,y) =: d_2((x,x'),(y,y')), \end{aligned}$$

which means that d is 1-Lipschitz with respect to d_2. In particular it is continuous.

The answer to both questions is yes. We saw in Problem Set 2.6 that d_1, d_2, d_3 are all comparable distances on the product $X \times X$. So any function which is (Lipschitz) continuous with respect to one of them is also (Lipschitz) continuous with respect to the others (possibly with a larger Lipschitz constant).

Solution of 3.3: Topological. Let U be an open set in Z, then $g^{-1}(U)$ is open in Y, since g is topologically continuous. Then also $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X, because also f is topologically continuous. Thus we showed that for all open sets $U \subset Z$, also $(g \circ f)^{-1}(U)$ is open. This means that $g \circ f : X \to Z$ is topologically continuous.

Sequential. Let (x_n) be a sequence in X converging to some $x \in X$. We need to show that $(g \circ f)(x_n)$ converges to $(g \circ f)(x)$ in Z.

Since f is continuous, $f(x_n)$ converges to $f(x)$ in Y. Similarly, as g is continuous, $g(f(x_n))$ converges to $g(f(x))$ in Z. Therefore, $g \circ f : X \to Z$ is sequentially continuous.

Solution of 3.4: Assume such continuous f exists, then it must be sequentially continuous. Lets us prove that this is never the case exhibiting two sequences converging to the origin along which f takes completely different values. Pick any sequence $t_k \downarrow 0$, then:

$$(t_k, t_k) \to (0,0) \text{ and } (t_k, -t_k) \to (0,0), \text{but } \lim_k f(t_k, t_k) = \tfrac{1}{2} \neq -\tfrac{1}{2} = \lim_k f(t_k, -t_k).$$

First of all let us show that if such g exists it is necessarily given by

$$g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } x = y = 0. \end{cases} \tag{1}$$

In fact $\lim_k g(1/k, 1/k) = \lim_k 1/k = 0$, so this can be the only value g takes at the origin.

Let us now check that the g defined by (1) is indeed continuous at the origin (continuity at the points outside the origin is obvious). Take any nonzero sequences $x_k, y_k \to 0$, then

$$|g(x_k, y_k)| = \frac{|x_k y_k|}{\sqrt{x_k^2 + y_k^2}} \leq \frac{x_k^2 + y_k^2}{\sqrt{x_k^2 + y_k^2}} \leq \sqrt{x_k^2 + y_k^2} \to 0.$$

Solution of 3.5: See Proposition 9.77 in the notes.

Solution of 3.6:

1. Not open, since $(0, 2+\epsilon^2, 0)$ lies outside E_1, but $(0, 2-\epsilon^2, 0)$ lies inside E_1. Not closed, since $(0, \epsilon^2, 0)$ lies inside E_1, but the limit point as $\epsilon \to 0$ does not: $(0, 0, 0, \notin E_1$. Not compact otherwise would be closed.

2. Notice that $E_2 = f^{-1}([1/4, \infty))$ with $f \colon \mathbb{R}^n \to \mathbb{R}$ give by $f(x) = \sin \|x\|$. Now f is continuous, $[1/4, \infty)$ is closed in $\mathbb{R}$, so E_2 is closed. On the other hand $\mathbb{R}^n$ is connected, so E_2 cannot be also open since:

   $$(\pi/2, 0, \ldots, 0) \in E_2, \quad (0, \ldots, 0) \notin E_2.$$

   Finally, E_2 cannot be compact because it is not bounded as $x_k := (2k\pi + \pi/2, 0, \ldots, 0) \in E_2$ for all $k \in \mathbb{N}$.

3. Define the continuous functions $g, f_n \colon \mathbb{R}^3 \to \mathbb{R}$ as

   $$f_n(x) := x_1^4 - \tfrac{1}{n} x_3^2 - x_2^2 - \tfrac{1}{n}, g(x) := 6 - x_1 - x_2.$$

   Then we can write

   $$E_3 := \bigcup_{n \geq 1} f_n^{-1}((-\infty, 0)) \cap g^{-1}((-\infty, 0)) = \Big\{ \bigcup_{n \geq 1} f_n^{-1}((-\infty, 0)) \Big\} \cap g^{-1}((-\infty, 0)),$$

   the term in curly braces is open (union of open) and so is E_3 (we intersect only once with another open set). Notice that $(1+t, -t, 0) \in E_3$ for all $t \geq 1$ so E_3 is unbounded and it cannot be compact (and it is nonempty). On the other hand $(0, 6, 1) \notin E_3$, so its complementary cannot be empty, hence E_3 is not closed ($\mathbb{R}^n$ is connected again).

4. Similarly, open using the continuous function

   $$(x, y) \mapsto x \cdot y - \|x\| \|y\|,$$

   unbounded by homogeneity $(x, y) \in E_4 \Leftrightarrow (tx, ty) \in E_4$. Not empty as $(e_n, e_n) \in E_4$, not everything as $(0, v) \notin E_4$.

5. Invertibility can be expressed as $E_5 = \det^{-1}(\mathbb{R} \setminus \{0\})$, since the determinant is a polynomial function of the entries it is continuous, so E_5 is open. Again is unbounded by homogeneity and not everything (not all matrices are invertible).

6. $E_6 = f^{-1}(\{0\})$ with $f(X) = \sum_{i,j} |X_{ij} - X_{ji}|^2$. So E_6 is closed. Cannot be open (not all matrices are symmetric, there is at least one symmetric matrix) nor compact (one can find symmetric matrices with arbitrarily large entries).
7. This is a finite set of "points" in $\mathbb{R}^{n\times n}$. Hence it is compact, closed and not open.
8. $E_8 = f^{-1}(-\infty, 6])$ with $f(x) = \max_{1\le i\le n} |x_i|$ so it is closed. It is not everything nor nothing, so it is not open. Furthermore it is compact since it is clearly bounded:

$$x \in E_8 \Rightarrow \|x\|^2 \le \sum_{i=1}^{n} 6^2 \le 100^n.$$

9. Since E_8 is unbounded so is E_9, so it cannot be compact. Being a product of and open and a compact set we expect it to be none of them. If it was open, then write it as union of open balls $\{B(x_i, r_i\}_{i\in I} \subset \mathbb{R}^{n+3}$. Then the balls

$$B' := B(x_i, r_i) \cap \{x_{n+1} = \ldots = x_{n+3}0\} \subset \mathbb{R}^3$$

would be open and contained in E_8, so E_8 would be open, contradiction.

Pick a sequence $\{x_k\} \subset E_3 \subset \mathbb{R}^n$ that converges to some $z \in \mathbb{R}^n \setminus E_3$. Then the sequence $x'_k := (x_k, 0, 0, 0) \in \mathbb{R}^{n+3}$ is convergent to $(z, 0, 0, 0)$. If E_9 was closed we would get $(z, 0, 0, 0) \in E_3 \times E_8$, hence $z \in E_3$, contradiction.
10. Being a intersection of a closed and a compact set it is compact and so closed. It is not empty as $(\pi/2, 0, \ldots, 0 \in E_{10}$, so it is not open.

Solution of 3.7:

(a) True. Let U be such a set, then pick any $z \in \partial U$ and let $f \in C(U)$ be defined by $f(x) := \|x - z\|^{-1}$. If U was compact f would be bounded, we can pick a sequence $x_k \in U$ such that $x_k \to z$, so $f(x_k) \to \infty$, contradiction.

We remark that ∂U is not empty otherwise $U = \overline{U}$ would be clopen, but it is not empty nor the whole $\mathbb{R}^n$ by assumption.

(b) True. Let U be such a set, pick $z \in \partial U$ and a sequence $x_k \in U$ such that $x_k \to z$. While $\{x_k\}$ is Cauchy (is convergent in $\mathbb{R}^n$) its limit point does not belong to U.

(c) True, it must be closed as subset of $\mathbb{R}^n$.

(d) False. A finite subset of $\mathbb{R}^n$ is always compact, thus complete, but $\mathbb{Q}$ is not complete, while being a countable union of points.

(f) True. A sequence in $\mathbb{R}^n$ converges if and only if it is Cauchy, so, for a set, containing its accumulation points or the limits of its Cauchy sequences is the same property.

Solution of 3.8:

(a) True, at least if your desk is in Zurich. The map (literally) maps Zurich to a piece of paper, so point which in reality are hundreds of meters apart are distant few millimeters in the map. So the Lipschitz constant is approximately 1/10000. Furthermore your desk is in Zurich, so the map sends the domain in itself (if you are in e-learning from Singapore then it it is false).

(b) False, $f(x) := \frac{x}{10}$ is a counterexample.

(c) True, Banach's fixed point theorem applies since $[0,1] \subset [0,2]$.

(d) False, take f that swaps $[0,1] \leftrightarrow [2,3]$ contracting them of a factor 1/10. In symbols:

$$f(x) := 2 + \tfrac{x}{10} \text{ for all } x \in [0,1], \quad f(x) := x - 2 + \tfrac{x}{10} \text{ for all } x \in [2,3].$$

(e) True. Take $\epsilon > 0$ and consider $f_\epsilon(x) := (1-\epsilon)f(x)$. Notice that $f_\epsilon([0,1]) \subset [0, 1-\epsilon] \subset [0,1]$. Also, $|f'_\epsilon(x)| < 1 - \epsilon$, so by the Mean Value Theorem it is a contraction (the interval is connected). So Banach's fixed point theorem gives $x_\epsilon \in [0,1]$ such that

$$x_\epsilon = f_\epsilon(x_\epsilon) = (1-\epsilon)f(x_\epsilon).$$

By compactness, there is $\epsilon_k \to 0$ and $x_0 \in [0,1]$ such that $x_{\epsilon_k} \to x_0$, so we can pass this equation to the limit and find that x_0 is a fixed point.

Actually, applying the intermediate value theorem to the function $g(x) - x$, one can show that every continuous function $g\colon [0,1] \to [0,1]$ has a fixed point.

Solution of 3.9:

(a) True, it is in fact Lipschitz continuous:

$$|(x+y) - (x'+y')| \le |x - x'| + |y - y'| \le 2\sqrt{|x-x'|^2 + |y-y'|^2},$$

where we used $a + b \le 2\sqrt{a^2 + b^2}$.

(b) False. Let $f(x,y) := xy$. Consider for some $\delta > 0$ fixed, the balls $B_t := B((t,t),\delta)$ centered at the point (t,t). The size of the interval $f(B_t)$ must go to infinity as $t \to \infty$, so it is impossible that it stays contained in $[-1+t^2, t^2+1] = B_1(f(t,t)) \subset \mathbb{R}$, violating the definition of uniform continuity with $\epsilon = 1$.

Check: $f(t, t \pm \delta) = t^2 \pm 2t\delta + \delta^2$, so $f(B_t) \subset \mathbb{R}$ is an interval (it is connected) that contains the interval $[t^2 - 2t\delta + \delta^2, t^2 + 2t\delta + \delta^2]$, whose length goes to infinity as $t \to \infty$ with δ fixed.

(c) True because of the Heine-Cantor Theorem 3.5.

(d) True because of the Heine-Cantor Theorem 3.5.