Questions marked with (*) are a bit more complex, you might want to skip them at the first read. Hints available in the next page.

4.1. BONUS PROBLEM. Consider the function $u\colon (x,y) \mapsto x^{\sin(y)}$, defined for $(x,y) \in (0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$. Compute $\partial_x u$ and $\partial_y u$.

4.2. Connected graphs. Let $U \subset \mathbb{R}^n$ be open and connected and let $f \in C^1(U, \mathbb{R}^m)$. Show that its graph

$$\Gamma_f := \{(x, f(x)) : x \in U\}$$

is a connected subset of $\mathbb{R}^n \times \mathbb{R}^m$.

4.3. *p*–norms. For $p \geq 1$ and $x \in \mathbb{R}^n$ define the $p-$norm of x as

$$|x|_p := \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}.$$

1. For $n = 2$ and $p = 1, 2, 10$ sketch the sets $\{x \in \mathbb{R}^2 : |x|_p \leq 1\}$.
2. For a given $x \in \mathbb{R}^n$, compute the limit $|x|_\infty := \lim_{p\to\infty} |x|_p$.
3. Using an appropriate inequality that you have seen in class, prove that

$$\Big(\sum_{i=1}^n a_i^{p-1} b_i\Big)^2 \leq \Big(\sum_{i=1}^n a_i^p\Big)\Big(\sum_{i=1}^n a_i^{p-2} b_i^2\Big),$$

   whenever a_i, b_i are n-tuples of positive numbers.
4. Fix $x, y \in \mathbb{R}^n$ and consider the function $f\colon [0,1] \to \mathbb{R}$ defined as

$$f(t) := |tx + (1-t)y|_p = \Big(\sum_{i=1}^n |tx_i + (1-t)y_i|^p\Big)^{1/p}. \tag{1}$$

   Show that f is convex. You may assume that the coordinates of x and y are all strictly positive and use the inequality of the previous point.
5. Deduce from the previous point that the triangular inequality holds, i.e.,

$$|x+y|_p \leq |x|_p + |y|_p \quad \text{for all } x, y \in \mathbb{R}^n.$$

6. What happens for $p \in (0,1)$?

4.4. *p*–means. For $x \in \mathbb{R}^n$ with positive coordinates and $p \neq 0$ define the p-mean as

$$\mu_p(x) := \Big(\frac{x_1^p + \ldots + x_n^p}{n}\Big)^{1/p}.$$

1. Compute the limits $p \to \pm\infty, p \to 0$ and define accordingly

$$\mu_{-\infty}(x), \quad \mu_0(x), \quad \mu_{+\infty}(x).$$

2. For any n-tuple of numbers $a_i > 0$ show that

$$\sum_{i=1}^{n} \frac{a_i}{a_1 + \ldots + a_n} \log(a_i) \geq \log\Big(\frac{a_1 \ldots + a_n}{n}\Big).$$

3. For a fixed x, show that the function $f\colon \mathbb{R} \to \mathbb{R}$, given by $f(t) := \mu_t(x)$, is continuous and increasing.

4. Prove the Arithmetic-Geometric inequality and Arithmetic-Quadratic inequality:

$$n(x_1 x_2 \cdots x_n)^{1/n} \leq x_1 + \ldots + x_n, \qquad (x_1 + \ldots + x_n)^2 \leq n(x_1^2 + \ldots + x_n^2).$$

5. (*) Is f continuously differentiable in the whole $\mathbb{R}$?

4.5. All norms are equivalent in $\mathbb{R}^n$. Let $|\cdot|$ denote the standard Euclidean norm in $\mathbb{R}^n$ and let $f\colon \mathbb{R}^n \to [0, \infty)$ be another norm (that is a function satisfying the properties of Definition 9.91).

1. Expressing x in a basis and using the “abstract” properties that f must have, show that there is a constant $C_1 > 0$ such that

$$f(x) \leq C_1 |x| \text{ for all } x \in \mathbb{R}^n.$$

2. Show that f is continuous in $\mathbb{R}^n$ (with respect to the standard distance of $\mathbb{R}^n$!).

3. Show that there is a number $c_2 > 0$ such that

$$f(x) \geq c_2 \text{ for all } |x| = 1.$$

4. Conclude that $f(x) \geq c_2 |x|$ for all $x \in \mathbb{R}^n$.

5. Show that if $\tilde{f}$ is yet another norm, then there is $C > 0$ such that

$$C^{-1} f(x) \leq \tilde{f}(x) \leq C f(x) \text{ for all } x \in \mathbb{R}^n.$$

4.6. Hilbert Schmidt norm of the composition. Take two linear functions $\phi\colon \mathbb{R}^d \to \mathbb{R}^n$ and $\psi\colon \mathbb{R}^n \to \mathbb{R}^m$, and denote with Φ, Ψ the matrices that represent them in the canonical basis’. Recall that the linear map $\psi \circ \phi\colon \mathbb{R}^d \to \mathbb{R}^m$ is represented in these basis’ by the matrix $\Psi \cdot \Phi$. Show that

$$\|\Psi \cdot \Phi\| \leq \|\Psi\| \|\Phi\|,$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm of a matrix (see 10.1.3 in the notes).

4.7. Mean value for vector-valued functions. Let $f \in C^1(\mathbb{R}, \mathbb{R}^m)$ for $m > 1$. Is it true that there is $t \in [0, 1]$ such that

$$f(1) - f(0) = Df_t(1) = \begin{bmatrix} f_1'(t) \\ \vdots \\ f_m'(t) \end{bmatrix}?$$

Prove it or provide a counterexample.

4.8. A directional derivative vanish. Let $u \in C^1(\mathbb{R}^n)$ and $\nu \in \mathbb{R}^n$. Show that

1. If $\partial_1 u \equiv 0$ then "u does not depend on x_1", more rigorously: there exists a unique function $v \in C^1(\mathbb{R}^{n-1})$ such that

$$u(x_1, \dots, x_n) = v(x_2, \dots, x_n) \text{ for all } x \in \mathbb{R}^n. \tag{2}$$

2. If $\partial_\nu u \equiv 0$ and $\nu \cdot e_1 \neq 0$ then "u is a function of $n-1$ variables", more rigorously: there exists a unique function $w \in C^1(\mathbb{R}^{n-1})$ such that

$$u(x_1, \dots, x_n) = w(x_2 - \tfrac{x_1 \nu_2}{\nu_1}, \dots, x_n - \tfrac{x_1 \nu_n}{\nu_1}) \text{ for all } x \in \mathbb{R}^n.$$

3. (*) What can we conclude if we assume only that $\partial_1 u = 0$ in an open connected subset $U \subset \mathbb{R}^n$?

Hints:

4.3.3 Use Cauchy–Schwartz and the fact that $a_i^{p-1} b_i = a_i^{p/2} \cdot a_i^{(p-2)/2} b_i$.

4.3.4 Show that $f''(t) \geq 0$, it is not the simplest derivative, but if you get it right the inequality in 4.2.3 will be just what you need to prove that it $f'' \geq 0$.

4.3.5 Write the convexity inequality between x, y and the middle point $(x+y)/2$. To get the general case from the one with positive coordinates observe the following. For $x \in \mathbb{R}^n$ denote with $\hat{x} := (|x_1|, \ldots, |x_n|)$, then

$$|x+y|_p \leq |\hat{x}+\hat{y}|_p, \quad |\hat{x}|_p = |x|_p, \quad \text{for all } x, y \in \mathbb{R}^n.$$

4.4.2 Combine the concavity of the $\log(\cdot)$ and the Cauchy–Schwarz inequality. You need to use the following little generalisation of the concavity inequality: for any concave function $f\colon \mathbb{R} \to \mathbb{R}$:

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$
$$\text{for all } x_i \in \mathbb{R}, 0 \leq \lambda_i \leq 1 \text{ with } \lambda_1 + \ldots + \lambda_n = 1.$$

4.4.3 It might be more convenient to work with $\log f(t)$, be careful with the computation, once again the inequality in 4.3.2 will be just what you need to prove that $f' \geq 0$.

4.5.2 Recall that from the definition it follows that $|f(x) - f(y)| \leq f(x-y)$... And that Lipschitz functions are always continuous.

4.5.3 Apply Weierstrass Theorem to f on $S := \{x \in \mathbb{R}^n : |x| = 1\}$ and use that f, being a norm, is nondegenerate.

4.5.5 If f is equivalent to $|\cdot|$ and $\tilde{f}$ is equivalent to $|\cdot|$ it follows by transitivity that f is equivalent to $\tilde{f}$...

4.6 First recall that $\|M\|^2$ is the sum of the squares of the entries of M. Notice that $(\Psi \cdot \Phi)^i_j$ (ith row and jth column) is the scalar product of Ψ^i and Φ_j which are vectors of $\mathbb{R}^n$. Apply Cauchy-Schwartz to each of them.

4.7 Try $f(t) = (\sin(2\pi t), \cos(2\pi t))$...

4.8.2 Apply the mean value theorem to $u(x + t\nu)$...

4.8.3 Consider the domain $U := \{(x,y) : x^2 < y < x^2 + 1\}$ and the function

$$u(x,y) := \begin{cases} \max\{0, y-2\}^2 & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$