Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems marked with (♡).

5.1. BONUS PROBLEM.

(a) Consider $f(x,y) := xy^2 e^{-x^2-y^2}$ for $(x,y) \in \mathbb{R}^2$. Find all the critical points of f, that is all the points where the gradient of f vanishes. (The point will be given if and only if all the numerical values you find are correct... so check your computations twice).

(b) We want to find a function $g \in C^1(\mathbb{R}^2)$ with the following directional derivative:

$$\partial_v g(x,y) = 2\cos(x^2 y) x v_2 + \cos(x^2 y) v_1^2 y \quad \text{for all } (x,y) \text{ and } (v_1,v_2) \in \mathbb{R}^2.$$

Give an explicit example of such a function g or prove that a function with these properties cannot exist.

5.2. Lagrange multipliers (♡). Consider the ball $U := \{(x,y,z) : x^2+y^2+z^2 < 1\}$ and the function

$$f(x,y,z) := 1 - x^2 - y^2 + 4x.$$

1. Motivate rigorously why f attains its absolute maximum and minimum in $\overline{U}$.
2. Find all the critical points of f which lie in the interior of U. Say whether $\sup_U f$ (or $\inf_U f$) is attained at some point in U.
3. Say whether $\max_{\partial U} f$ and $\min_{\partial U}$ exist.
4. With the method of Lagrange multipliers, find the possible critical points for the constrained problem

$$\max\{f(x,y,z) : (x,y,z) \in \partial U\},$$

and same for min.
5. Among all the points you have found clarify at which points the maximum/minimum of f is attained.

5.3. Multiple choice (♡). Mark all and only the statements which are true

(a) As $|x| \to 0$, if $f(x) = x_2 x_1^2 + O(x_1^2)$, then $f(x) = O(|x|^2)$.

(b) As $x \to 0$, if $f(x) = x_2 x_1^2 + O(x_1^3)$, then $f(x) = O(x_1^3)$.

(c) As $|x| \to 0$, if $f(x) = x_2 x_1 + O(|x|^2)$, then f is differentiable at $x = 0$.

(d) As $x \to 0$, if $f(x) = x_2 x_1 + O(|x|^2)$, then f is twice differentiable around $x = 0$.

(e) As $|x| \to 0$, if $f(x) = x_2 x_1 + O(|x|^3)$, then f is twice differentiable around $x = 0$ and $\partial_{11} f(0) = 0$.

5.4. Computation of derivatives . For each of the following functions compute the directional derivative in the general direction v at a general point x, that is $\partial_v f(x)$. Then, say where the function is differentiable and give the size of its Jacobi matrix.

1. (♡) $x_1/x_2, e^{-x_1/x_2}, e^{x_1x_2}\sin(x_1+x_2^2), \frac{x_1^2}{x_1^2+x_2^2}, \frac{x_1^2x_2}{x_1^2+x_2^2}, \frac{x_1^2x_2}{x_1^2+x_2^4}$ for $x \in \mathbb{R}^2 \setminus \{0\}$.
2. (♡) $|x|, |x|^\alpha, x/|x|, g(|x|), x\cdot e, g(x\cdot e)$ for $x \in \mathbb{R}^n \setminus \{0\}, \alpha \in \mathbb{R}, e \in \mathbb{R}^n$ and $g \in C^1(\mathbb{R})$.
3. $ax, x^T, \mathrm{Tr}(x), x^2, x^3, \mathrm{Tr}(x^2)$ for $x, a \in \mathbb{R}^{n\times n}$,
4. (*) x^{-1}, x^{-2} for $x \in \mathbb{R}^{n\times n}$ with $\det x \neq 0$.

5.5. Multiple choice. Assume $g \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\mathbb{R}^n)$ is such that

$$f(x) = x_1 + x_2x_1 + O(|x|^4)\text{as } x \to 0.$$

We want to compute the Taylor expansion of $(g \circ f)$ at $x = 0$ of the highest possible order with the information we have.

We can ask for the value $g^{(k)}(t)$ for arbitrary $k \in \mathbb{N}, t \in \mathbb{R}$, but we have to pay 5 CHF each time.

For a general g, which is the degree of the best (i.e., highest-order) Taylor polynomial we can compute? How expensive will it be to compute it?

(a) degree 3 and 20 CHF

(b) degree 2 and 15 CHF

(c) degree 2 and 10 CHF

(d) degree 3 and 15 CHF

5.6. Polynomials. Let $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$ be polynomials of n variables, assume that there are positive numbers M, σ such that

$$|P(x) - Q(x)| \leq M|x|^\sigma \text{ for all } |x| \leq 1.$$

Show that P and Q have the same coefficients of order smaller than σ. That is, if we write

$$P(X) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha, \qquad Q(X) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha X^\alpha,$$

then $a_\alpha = b_\alpha$ for all $|\alpha| < \sigma$.

Hints:

5.3 Revise Corollary 10.33.

5.4 In order to prove that a function **is** differentiable at 0 you have two tools

- the Definition of differentiability;
- the sufficient condition of Theorem 10.11.

In order to prove that a function **is not** differentiable at 0 you have two tools

- Show that the necessary condition of Proposition 10.9 does not hold;
- Show that for some C^1 curve $\gamma\colon (-\delta, \delta) \to \mathbb{R}^n$ the function $(f \circ \gamma)(t)$ is **not** differentiable at $t = 0$. This could be simpler as you have to study a function of only one variable.

5.4.3 These functions are C^∞ so try to compute them on $x + tv$ and find an expansion in powers of t... If you find something it must be the Taylor polynomial by Corollary 10.33. From the Taylor polynomial it is immediate to read off the derivatives. For example

$$(x + tv)^2 = (x + tv)(x + tv) = x^2 + txv + tvx + t^2 = x^2 + (xv + vx)t + O(t^2)...$$