Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems marked with (♡).

5.1. BONUS PROBLEM.

(a) Consider $f(x, y) := xy^2e^{-x^2-y^2}$ for $(x, y) \in \mathbb{R}^2$. Find all the critical points of f, that is all the points where the gradient of f vanishes. (The point will be given if and only if all the numerical values you find are correct... so check your computations twice).

(b) We want to find a function $g \in C^1(\mathbb{R}^2)$ with the following directional derivative:

$$\partial_v g(x, y) = 2\cos(x^2y)xv_2 + \cos(x^2y)v_1^2y \quad \text{for all } (x, y) \text{ and } (v_1, v_2) \in \mathbb{R}^2.$$

Give an explicit example of such a function g or prove that a function with these properties cannot exist.

5.2. Lagrange multipliers (♡). Consider the ball $U := \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ and the function

$$f(x, y, z) := 1 - x^2 - y^2 + 4x.$$

1. Motivate rigorously why f attains its absolute maximum and minimum in $\overline{U}$.
2. Find all the critical points of f which lie in the interior of U. Say whether $\sup_U f$ (or $\inf_U f$) is attained at some point in U.
3. Say whether $\max_{\partial U} f$ and $\min_{\partial U}$ exist.
4. With the method of Lagrange multipliers, find the possible critical points for the constrained problem

$$\max\{f(x, y, z) : (x, y, z) \in \partial U\},$$

and same for min.

5. Among all the points you have found clarify at which points the maximum/minimum of f is attained.

5.3. Multiple choice (♡). Mark all and only the statements which are true

(a) As $|x| \to 0$, if $f(x) = x_2x_1^2 + O(x_1^2)$, then $f(x) = O(|x|^2)$.

(b) As $x \to 0$, if $f(x) = x_2x_1^2 + O(x_1^3)$, then $f(x) = O(x_1^3)$.

(c) As $|x| \to 0$, if $f(x) = x_2x_1 + O(|x|^2)$, then f is differentiable at $x = 0$.

(d) As $x \to 0$, if $f(x) = x_2x_1 + O(|x|^2)$, then f is twice differentiable around $x = 0$.

(e) As $|x| \to 0$, if $f(x) = x_2x_1 + O(|x|^3)$, then f is twice differentiable around $x = 0$ and $\partial_{11}f(0) = 0$.

5.4. Computation of derivatives . For each of the following functions compute the directional derivative in the general direction v at a general point x, that is $\partial_v f(x)$. Then, say where the function is differentiable and give the size of its Jacobi matrix.

1. (♡) $x_1/x_2, e^{-x_1/x_2}, e^{x_1x_2}\sin(x_1+x_2^2), \frac{x_1^2}{x_1^2+x_2^2}, \frac{x_1^2x_2}{x_1^2+x_2^2}, \frac{x_1^2x_2}{x_1^2+x_2^4}$ for $x \in \mathbb{R}^2 \setminus \{0\}$.
2. (♡) $|x|, |x|^\alpha, x/|x|, g(|x|), x\cdot e, g(x\cdot e)$ for $x \in \mathbb{R}^n \setminus \{0\}, \alpha \in \mathbb{R}, e \in \mathbb{R}^n$ and $g \in C^1(\mathbb{R})$.
3. $ax, x^T, \mathrm{Tr}(x), x^2, x^3, \mathrm{Tr}(x^2)$ for $x, a \in \mathbb{R}^{n\times n}$,
4. (*) x^{-1}, x^{-2} for $x \in \mathbb{R}^{n\times n}$ with $\det x \neq 0$.

5.5. Multiple choice. Assume $g \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\mathbb{R}^n)$ is such that

$$f(x) = x_1 + x_2x_1 + O(|x|^4)\text{as } x \to 0.$$

We want to compute the Taylor expansion of $(g \circ f)$ at $x = 0$ of the highest possible order with the information we have.

We can ask for the value $g^{(k)}(t)$ for arbitrary $k \in \mathbb{N}, t \in \mathbb{R}$, but we have to pay 5 CHF each time.

For a general g, which is the degree of the best (i.e., highest-order) Taylor polynomial we can compute? How expensive will it be to compute it?

(a) degree 3 and 20 CHF

(b) degree 2 and 15 CHF

(c) degree 2 and 10 CHF

(d) degree 3 and 15 CHF

5.6. Polynomials. Let $P, Q \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials of n variables, assume that there are positive numbers M, σ such that

$$|P(x) - Q(x)| \leq M|x|^\sigma \text{ for all } |x| \leq 1.$$

Show that P and Q have the same coefficients of order smaller than σ. That is, if we write

$$P(X) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha, \qquad Q(X) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha X^\alpha,$$

then $a_\alpha = b_\alpha$ for all $|\alpha| < \sigma$.

Hints:

5.3 Revise Corollary 10.33.

5.4 In order to prove that a function **is** differentiable at 0 you have two tools

- the Definition of differentiability;
- the sufficient condition of Theorem 10.11.

In order to prove that a function **is not** differentiable at 0 you have two tools

- Show that the necessary condition of Proposition 10.9 does not hold;
- Show that for some C^1 curve $\gamma\colon (-\delta, \delta) \to \mathbb{R}^n$ the function $(f \circ \gamma)(t)$ is **not** differentiable at $t = 0$. This could be simpler as you have to study a function of only one variable.

5.4.3 These functions are C^∞ so try to compute them on $x + tv$ and find an expansion in powers of t... If you find something it must be the Taylor polynomial by Corollary 10.33. From the Taylor polynomial it is immediate to read off the derivatives. For example

$$(x + tv)^2 = (x + tv)(x + tv) = x^2 + txv + tvx + t^2 = x^2 + (xv + vx)t + O(t^2)...$$

5. Solutions

Solution of 5.1:

(a) Solving the system

$$\begin{cases} 0 = \partial_x f = (1-2x^2)y^2 e^{-x^2-y^2} \\ 0 = \partial_y f = 2(y-y^3)x e^{-x^2-y^2} \end{cases}$$

we find that critical points are:

$$\{(s,0) : s \in \mathbb{R}\}, \quad (\tfrac{1}{\sqrt{2}}, 1), \quad (-\tfrac{1}{\sqrt{2}}, 1), \quad (\tfrac{1}{\sqrt{2}}, -1), \quad (-\tfrac{1}{\sqrt{2}}, -1).$$

(b) Such a function cannot exist. By Proposition 10.9, $\partial_v g(x,y)$ must be a linear function of v when (x,y) are kept fixed, and this is not the case because of the term v_1^2, which is nonlinear.

Solution of 5.2:

1. Since f is continuous and $\overline{U}$ is compact (closed and bounded), it must take within $\overline{U}$ its maximum and minimum value.
2. We are asked to find all $(x,y,z) \in U$ such that $\nabla f(x,y,z) = 0$, this is equivalent to solve the system
$$\begin{cases} 0 = \partial_x f = 4 - 2x \\ 0 = \partial_y f = -2y \\ 0 = \partial_z f = 0 \\ x^2 + y^2 + z^2 < 1 \end{cases}$$
which has no solutions since the first equation gives $x = 2$, contradicting the last condition. Not having critical points in U, by Proposition 11.4 we infer that f does not attain inside U its sup nor inf.
3. Since ∂U is compact, they must exist by the same reasoning of point 1.
4. Let's call $g(x,y,z) := x^2+y^2+z^2-1$. By Proposition 11.5, any critical point (relative to the minimization problem constrained to the boundary) solves the system
$$\begin{cases} 0 = \lambda_0 \partial_x f(x,y,z) + \lambda_1 \partial_x g(x,y,z) \\ 0 = \lambda_0 \partial_y f(x,y,z) + \lambda_1 \partial_y g(x,y,z) \\ 0 = \lambda_0 \partial_z f(x,y,z) + \lambda_1 \partial_z g(x,y,z) \\ \lambda_0^2 + \lambda_1^2 = 1 \\ g(x,y,z) = 0 \end{cases}$$

which becomes

$$\begin{cases} 0 = \lambda_0(4-2x) + 2\lambda_1 x \\ 0 = \lambda_0(-2y) + 2\lambda_1 y \\ 0 = 2\lambda_1 z \\ \lambda_0^2 + \lambda_1^2 = 1 \\ x^2+y^2+z^2 = 1 \end{cases},$$

Notice that if $\lambda_0 = 0$ then we would get $x = y = z = 0$ which does not lie on ∂U, so setting $\lambda := \lambda_1/\lambda_0$ we have the equivalent system

$$\begin{cases} 0 = 2 - x + \lambda x = 2 + (\lambda - 1)x \\ 0 = -y + \lambda y = (\lambda - 1)y \\ 0 = \lambda z \\ x^2+y^2+z^2 = 1 \end{cases}$$

observe that if $\lambda = 0$ then we would have $x = -2$ which contradicts $(x, y, z) \in \partial U$. So we find $z = 0$ and the equivalent system:

$$\begin{cases} 0 = 2 + (\lambda - 1)x \\ 0 = (\lambda - 1)y \\ x^2+y^2 = 1 \end{cases}$$

which forces $(x, \lambda) \in \{(-1, 3), (1, -1)\}$ and $y = 0$. Summing up the critical (x, y, z, λ) are

$$(1, 0, 0, -1) \quad (-1, 0, 0, 3).$$

5. Since we showed that $\max_{\overline{U}}$ is attained at some point $P \in \overline{U}$ and that $P \notin U$, we must have $P \in \partial U$ and, by Proposition 11.5, that P is critical for the constrained problem
$$\max_{\partial U} f.$$
Same reasoning for $\min_{\overline{U}} f$. So the only two candidates are
$$(1, 0, 0) \quad (-1, 0, 0).$$
We check:
$$f(1, 0, 0) = +4 \quad f(-1, 0, 0) = -4.$$
So the point $(1, 0, 0)$ is an absolute maximum for $f|_{\overline{U}}$ and the point $(-1, 0, 0)$ is an absolute minimum for $f|_{\overline{U}}$.

Solution of 5.3:

(a) True. As $x \to 0$, we have $x_2 x_1^2 = O(|x|^3) \leq O(|x|^2)$ and $O(x_1^2) \leq O(|x|^2)$.

(b) False. The function $x_2x_1^2$ is **not** $O(x_1^3)$, in fact the ratio

$$\frac{x_2x_1^2}{x_1^3} = \frac{x_2}{x_1}$$

is unbounded in any neighbourhood of the origin.

(c) True. Just apply the definition of differentiability with $Df(0) = 0$.

(d) False. Take for example $f(x) = \max\{0, x_1\}^2$. Notice that saying that $f(x) = x_2x_1 + O(|x|^2)$, is exactly the same as saying that $f(x) = O(|x|^2)$.

(e) False. This would be true if we had the *a priori* assumption that $f \in C^2$ in a small ball around the origin. Without this assumptions it is easy to construct counterexamples as in Analysis I, for example

$$f(x) = x_1x_2 + |x|^4 \sin(|x|^{-100}),$$

in such a way that $|\partial_1 f(x)| \to \infty$ as $|x| \to 0$, so $\partial_1 f(x)$ is not even continuous at $x = 0$, let alone differentiable.

Solution of 5.4:

1. To compute the directional derivatives of the given functions with respect to the direction vector $v = (v_1, v_2)$, we use the definition of directional derivative:

$$\partial_v f(x) = v \cdot \nabla f(x) = v_1\partial_1 f(x) + v_2\partial_2 f(x),$$

where $\nabla f(x)$ is the gradient of the function $f(x)$.

- $f(x) = \frac{x_1}{x_2}$. The gradient of f is $\nabla f(x) = \left(\frac{1}{x_2}, -\frac{x_1}{x_2^2}\right)$, so, the directional derivative is

$$\partial_v f(x) = \frac{v_1}{x_2} - \frac{v_2x_1}{x_2^2}.$$

This function is differentiable outside the origin by Theorem 10.11. This function is not continuous at the origin, so it cannot be differentiable there.

- $f(x) = e^{-\frac{x_1}{x_2}}$. The gradient of f is $\nabla f(x) = \left(\frac{1}{x_2}e^{-\frac{x_1}{x_2}}, \frac{x_1}{x_2^2}e^{-\frac{x_1}{x_2}}\right)$, so, the directional derivative is

$$D_{\mathbf{v}} f(x) = \frac{v_1}{x_2}e^{-\frac{x_1}{x_2}} + \frac{v_2x_1}{x_2^2}e^{-\frac{x_1}{x_2}}$$

So f is differentiable outside the origin by Theorem 10.11. Since f is not continuous at the origin, so it cannot be differentiable there.

- $f(x) = e^{x_1x_2}\sin(x_1 + x_2^2)$. The gradient of f is

$$\nabla f(x) = \left(x_2e^{x_1x_2}\sin(x_1 + x_2^2), x_1e^{x_1x_2}\sin(x_1 + x_2^2) + 2x_2e^{x_1x_2}\cos(x_1 + x_2^2)\right).$$

So, the directional derivative is:

$$\begin{aligned}\partial_v f(x) &= v_1x_2e^{x_1x_2}\sin(x_1 + x_2^2) \\ &\quad + v_2\left(x_1e^{x_1x_2}\sin(x_1 + x_2^2) + 2x_2e^{x_1x_2}\cos(x_1 + x_2^2)\right)\end{aligned}$$

So f is differentiable everywhere by Theorem 10.11.

- $f(x) = \frac{x_1^2}{x_1^2+x_2^2}$. The gradient of f is $\nabla f(x) = \left(\frac{2x_1}{x_1^2+x_2^2}, \frac{-2x_1x_2}{(x_1^2+x_2^2)^2}\right)$, so, the directional derivative is:
$$\partial_v f(x) = \frac{2x_1v_1}{x_1^2+x_2^2} - \frac{2x_1x_2v_2}{(x_1^2+x_2^2)^2}.$$
So f is differentiable outside the origin by Theorem 10.11. Since f is not continuous at the origin, so it cannot be differentiable there.

- $f(x) = \frac{x_1^2x_2}{x_1^2+x_2^2}$. The gradient of f is $\nabla f(x) = \left(\frac{2x_1x_2}{x_1^2+x_2^2}, \frac{x_1^3-x_1x_2^2}{(x_1^2+x_2^2)^2}\right)$, so, the directional derivative is:
$$\partial_v f(x) = \frac{2x_1x_2v_1}{x_1^2+x_2^2} + \frac{(x_1^3-x_1x_2^2)v_2}{(x_1^2+x_2^2)^2}.$$
So f is differentiable outside the origin by Theorem 10.11. Since f is not continuous at the origin, so it cannot be differentiable there.

- $f(x) = \frac{x_1^2x_2}{x_1^2+x_2^4}$. The gradient of f is $\nabla f(x) = \left(\frac{2x_1x_2}{x_1^2+x_2^4}, \frac{x_1^3-3x_1x_2^4}{(x_1^2+x_2^4)^2}\right)$, so, the directional derivative is:
$$\partial_v f(x) = \frac{2x_1x_2v_1}{x_1^2+x_2^4} + \frac{(x_1^3-3x_1x_2^4)v_2}{(x_1^2+x_2^4)^2}.$$
So f is differentiable outside the origin by Theorem 10.11. Since f is not continuous at the origin, so it cannot be differentiable there.

2. We denote with $\langle\cdot,\cdot\rangle$ the standard scalar product of $\mathbb{R}^n$.

- $|x|$: We compute
$$\partial_v|x| = \sum_{i=1}^n v_i\partial_i(\sqrt{|x|^2}) = \sum_{i=1}^n \frac{v_i}{2|x|}\partial_i(|x|^2) = \sum_{i=1}^n \frac{v_ix_i}{|x|} = \frac{\langle x,v\rangle}{|x|}.$$
$|x|$ is differentiable outside the origin. It is **not** differentiable at $x=0$ as can be seen by contradiction. Assume that for some $\xi\in\mathbb{R}^n$ we had
$$|x| = \langle x,\xi\rangle + o(|x|) \text{ as } |x|\to 0.$$
Then testing this along the lines $x=\pm t\xi$ for $t\downarrow 0$ we find that necessarily $\xi=0$, so $|x| = o(|x|)$, which is impossible.

- $|x|^\alpha$: We compute
$$\begin{aligned}\frac{\partial|x|^\alpha}{\partial v} &= \frac{\partial}{\partial v}(|x|^\alpha) = \frac{\partial}{\partial v}\left((x_1^2+x_2^2+\ldots+x_n^2)^{\alpha/2}\right)\\ &= \frac{\alpha}{2}(x_1^2+x_2^2+\ldots+x_n^2)^{\alpha/2-1}\frac{\partial}{\partial v}(x_1^2+x_2^2+\ldots+x_n^2) = \alpha|x|^{\alpha-2}\langle x,v\rangle\end{aligned}$$
For $\alpha>1$ we have
$$\lim_{|x|\to 0}\left|\frac{\partial|x|^\alpha}{\partial v}\right| = \lim_{|x|\to 0} O(|x|^{\alpha-1}) = 0,$$
so by Theorem 10.11, $|x|^\alpha$ is of class $C^1(\mathbb{R}^n)$. In the case $\alpha\le 1$ we do not have differentiability at the origin.

- $\frac{x}{|x|}$: using Leibnitz and the previous identity

$$\partial_v(x/|x|) = \partial_v(x)|x|^{-1} + x\partial_v(|x|^{-1}) = \frac{v|x|^2 - \langle x, v\rangle x}{|x|^3}.$$

  Since $x/|x|$ is not continuous at $x = 0$ it will not be differentiable either.

- $g(|x|)$: By chain rule

$$\partial_v g(|x|) = g'(x)\partial_v|x| = g'(|x|)\frac{\langle x, v\rangle}{|x|}$$

  A reasoning similar to the first two above gives that g is differentiable at $x = 0$ if and only if $\lim_{t\to 0^+} g'(t) = 0$.

- $x \cdot e$, being a linear function, it coincides with its differential. Let's check it:

$$\frac{\partial(x\cdot e)}{\partial v} = \frac{\partial}{\partial v}(x_1e_1 + x_2e_2 + \ldots + x_ne_n) = e \cdot v.$$

- $g(x\cdot e)$: By the chain rule:

$$\frac{\partial g(x\cdot e)}{\partial v} = g'(x)\frac{\partial(x\cdot e)}{\partial v} = g'(x\cdot e)(e\cdot v)$$

  and it is of class C^1 being a composition of C^1 functions.

3. The maps ax, a^T and $\mathrm{Tr}(x)$ are linear so their differential is easily computed using the definition

$$\partial_v(ax) = av, \quad \partial_v(x^T) = v^T \text{ and } \partial_v(\mathrm{Tr}(x)) = \mathrm{Tr}(v).$$

For x^2 it is convenient to use Proposition 10.33 (it is of class C^2 since the entries of x^2 are polynomials of the entries of x).

$$(x+tv)^2 = (x+tv)(x+tv) = x^2 + txv + tvx + t^2 = x^2 + (xv + vx)t + O(t^2), \quad (1)$$

and so we must have

$$\partial_v x^2 = xv + vx \qquad \text{(which is \textbf{not} } 2xv \text{ in general!!!).}$$

Similarly for x^3 one finds

$$\partial_v x^3 = x^2v + xvx + vx^2.$$

If we apply the trace we can commute it with the derivative, so one can guess

$$\partial_v \mathrm{Tr}(x^2) = \mathrm{Tr}(xv + vx) = 2\mathrm{Tr}(xv),$$

one can also check it tracing (1)

$$\mathrm{Tr}((x+tv)^2) = \mathrm{Tr}(x^2) + 2t\mathrm{Tr}(xv) + t^2\mathrm{Tr}(v^2) = \mathrm{Tr}(x^2) + 2t\mathrm{Tr}(xv) + O(t^2).$$

4. Define $U := \{x \in \mathbb{R}^{n\times n} : \det(x) \neq 0\}$ and notice that it is open (why?). We start observing that the function $\iota\colon x \mapsto x^{-1}$ is smooth (i.e., C^∞) from U in itself, indeed the formula for the entries of the inverse matrix is explicit and algebraic as long as we are in U [1] Thus we known that there is some $\partial_v \iota(x) \in \mathbb{R}^{n\times n}$ such that

$$(x + tv)^{-1} = x^{-1} + t\partial_v \iota(x) + O(t^2), \quad \text{as } t \to 0.$$

We use again the algebraic properties of the inverse:

$$\begin{aligned}\mathbf{1}_{n\times n} &= (x+tv)^{-1}(x+tv) = (x+tv)(x^{-1} + t\partial_v\iota(x) + O(t^2)) \\ &= \mathbf{1}_{n\times n} + tvx^{-1} + tx\partial_v\iota(x) + O(t^2).\end{aligned}$$

Simplifying we find

$$vx^{-1} + x\partial_v\iota(x) = O(t),$$

so the only possibility is that

$$vx^{-1} + x\partial_v\iota(x) = 0,$$

which finally leads to

$$\partial_v\iota(x) = \partial_v(x^{-1}) = -x^{-1}vx^{-1}.$$

Now we can compute $\partial_v(x^{-2})$. We call $\sigma\colon U \to U$ the map $x \mapsto x^2$ and so that $x^{-2} = \sigma \circ \iota(x)$. Recall that we showed $\partial_w\sigma(y) = yw + wy$. We apply the chain rule

$$\partial_v(\sigma\circ\iota)(x) = \partial_{\partial_v\iota(x)}\sigma(\iota(x)) = \iota(x)\cdot\partial_v\iota(x) + \partial_v\iota(x)\cdot\iota(x) = -x^{-2}vx^{-1} - x^{-1}vx^{-2}.$$

Solution of 5.5: The answer is (a). Since f is given up to a $O(|x|^4)$ error, there is no hope to find any Taylor polynomial of degree larger than 3 (for example g could be the identity). Now Taylor expanding $(g\circ f)(x)$ we find

$$\begin{aligned}(g\circ f)(x) &= g(x_1 + x_1x_2 + O(|x|^4)) \\ &= g(0) + g'(0)(x_1 + x_1x_2 + O(|x|^4)) + \tfrac{g''(0)}{2}(x_1 + x_1x_2 + O(|x|^4))^2 \\ &\quad + \tfrac{g'''(0)}{6}(x_1 + x_1x_2 + O(|x|^4))^3 + O((x_1 + x_1x_2 + O(|x|^4))^4) \\ &= g(0) + g'(0)(x_1 + x_1x_2) + \tfrac{g''(0)}{2}(x_1^2 + 2x_1^2x_2) + \tfrac{g'''(0)}{6}x_1^3 + O(|x|^4).\end{aligned}$$

So we need the numbers $g(0), g'(0), g''(0), g'''(0)$ for which we need to pay 20 CHF. Notice that it would have been cheaper if we had the different assumption

$$f(x) = x_2x_1 + O(|x|^4) \text{ as } x \to 0.$$

[1] That formula is complicated and we do not want to use it directly, but it is useful to known that it exists and that is it is of the form

$$(x^{-1})_{ij} = \frac{\text{polynomial in the entries of } x}{\det x},$$

thus it is smooth as long as $\det(x) \neq 0$, that is to say $x \in U$.

Solution of 5.6: Without loss of generality, we can assume $Q = 0$ and that P has degree $N < \sigma$. Then we must have

$$\lim_{r\downarrow 0} \frac{|P(rx)|}{r^m} \leq \limsup_{r\downarrow 0} Mr^{\sigma-m} = 0 \text{ for all } m \leq N.$$

Assume that P is nonzero and take the shortest multiindex $\gamma \in \mathbb{N}^n$ such that there is $a_\gamma \neq 0$, so we have

$$\begin{aligned} |P(rx)| &\geq r^{|\gamma|}\Big| \sum_{\alpha\in\mathbb{N}^n, |\alpha|=|\gamma|} a_\alpha x^\alpha \Big| - r^{|\gamma|+1}\Big| \sum_{\alpha\in\mathbb{N}^n, N\geq|\alpha|>|\gamma|} a_\alpha r^{|\alpha|-|\gamma|-1} x^\alpha \Big| \\ &\geq r^{|\gamma|}\Big| \sum_{\alpha\in\mathbb{N}^n, |\alpha|=|\gamma|} a_\alpha x^\alpha \Big| - Cr^{|\gamma|+1}. \end{aligned}$$

Since $|\gamma| \leq N$, (**??**) entails $\left| \sum_{\alpha\in\mathbb{N}^n, |\alpha|=|\gamma|} a_\alpha x^\alpha \right| = 0$. But x was arbitrary and this means that all the a_α with $|\alpha| = |\gamma|$ are zero, which contradicts the choice of γ.