Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems/subquestions marked with (♡).

6.1. BONUS PROBLEM. Let $f \in C^2(\mathbb{R}^n)$ be a convex function.

(a) Show that $z \in \mathbb{R}^n$ is a critical point of f if and only if z is a global minimizer.

(b) Provide an example of such an f in some $\mathbb{R}^n$ with $n > 1$, which is always nonnegative, but does not have a minimum point. That is to say

$$f(x) > \inf_{\mathbb{R}^n} f \geq 0 \text{ for all } x \in \mathbb{R}^n.$$

You can use all the Theorems seen in class.

6.2. The signature of a 2 × 2 matrix. Despite the definition, it is not necessary to compute the eigenvalues of a matrix to find its signature[1]. Prove that for a 2×2 matrix M we have the following simple rule to determine the signature in terms of the $\det M$ and $\mathrm{Tr} M$:

- If $\det M > 0, \mathrm{Tr} M > 0$ then M is positive definite,
- If $\det M > 0, \mathrm{Tr} M < 0$ then M is negative definite,
- If $\det M < 0$ then M is indefinite,
- If $\det M = 0$, then M is degenerate.

6.3. Isoperimetric triangles. Among all the triangles with perimeter equal to 2, find the ones with the largest area. You may give for granted Heron's formula, which gives the area of a triangle in terms of the length of its sizes x, y, z:

$$A = \sqrt{p(p-x)(p-y)(p-z)}, \qquad \text{with } 2p := x + y + z,$$

so that in our case $p = 1$.

6.4. Barycenter (♡). Let $y_1, \dots, y_k \in \mathbb{R}^n$ be given. Show that there is exactly one point for which

$$f(x) = \|x - y_1\|^2 + \cdots + \|x - y_k\|^2, \quad x \in \mathbb{R}^n$$

is minimal and determine this point.

6.5. Linear regression I (♡). You study the house market in Zürich over a year in which N houses are sold. You keep track of the size of the houses $x_1, \dots, x_N$ and the respective sale prices $y_1, \dots, y_N$. Now you would like would like to find "the" function $f : \mathbb{R} \to \mathbb{R}$ that gives

$$\text{sale price} = f(\text{size of the house}),$$

[1] Ask ChatGPT about the Principal Minor Theorem

and you make the (not unreasonable) assumption that f is affine, i.e., $f_{a,b}(x) = ax + b$ for some coefficients $a, b \in \mathbb{R}$. Among all such functions find (in terms of the data you collected) the value of the parameters a, b that minimizes the average quadratic error

$$E(a,b) := \sum_{i=1}^{N} (y_i - f_{a,b}(x_i))^2, \qquad a, b \in \mathbb{R}.$$

6.6. Convex functions (♡). Decide whether the following functions f_i are convex in the convex domain $U_i \subset \mathbb{R}^n$. Try to find, in each case, the simplest argument, you can almost always avoid lengthy computations.

1. $f_1(x,y) = x^2 + y^2 - 4y$ defined in $U_1 = \mathbb{R}^2$
2. $f_2(x,y) = x^2 + y^2 - y^4$ defined in $U_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{10000}\}$
3. $f_3(x,y) = x^2 + y^2 - 4xy$ defined in $U_3 = \mathbb{R}^2$
4. $f_4(x,y) = x^2 + y^2 - 4xy$ defined in $U_4 = \{(x,y) \in \mathbb{R}^2 : 0 < 10x < |y|\}$
5. $f_5(x) = \phi(g(x)), x \in U_5$ where $g \in C^2(U_5)$ is any convex function in $U_5 \subset \mathbb{R}^n$ and $\phi \in C^2(\mathbb{R})$ is any convex and increasing function.
6. $f_6(x,y) = (1 + x^2 + y^2)^{1/2}$ defined in $U_6 = \mathbb{R}^2$
7. $f_7(x,y) = -(1 + x^2 + y^2)^{-1/2}$ defined in $U_7 = \mathbb{R}^2$
8. $f_8(x) = \sum_{i=1}^{n} |x_i|^p$ in $U_8 = \mathbb{R}^n$, where $p \geq 1$ is some fixed exponent.
9. $f_9(x) = \max\{\phi(x), \psi(x)\}$ where $\phi, \psi \in C(U_9)$ are any pair of convex functions defined in some open set $U_6 \subset \mathbb{R}^n$.
10. $f_{10}(x) = |x|$ defined in $U_{10} = \mathbb{R}^n$.
11. $f_{11}(x) = \phi(|x|)$ in $U_{11} = B_1 \subset \mathbb{R}^n$, where $\phi \in C(\mathbb{R})$ is any convex function.

6.7. Multiple choice. Among the following statements about convex functions mark those (and only those) which are always true.

(a) If $f \in C^1(U)$ is convex in some open convex set $U \subset \mathbb{R}^n$ and f has a local maximum at $z \in U$, then $\nabla f \equiv 0$ in U.

(b) If $f \in C^1(U)$ is convex in some open convex set $U \subset \mathbb{R}^n$ and f has a global maximum at $z \in U$, then $\nabla f \equiv 0$ in U.

(c) Assume $f_n \in C^2(\mathbb{R})$ is a sequence of convex functions that converge pointwise to some $f\colon \mathbb{R} \to \mathbb{R}$. Is f necessarily convex?

(d) There exists a convex function $f \in C^\infty(\mathbb{R}^2)$ such that

$$f(x) = 1 - 2x_1 + x_2^3 + O(|x|^4) \text{ as } |x| \to 0.$$

(e) There exists a convex function $f \in C^\infty(\mathbb{R}^2)$ such that

$$f(x) = 1 - 2x_1 + x_2^4 + O(|x|^4) \text{ as } |x| \to 0.$$

(f) A convex set is not necessarily connected.

6.8. Multiple choice. The Hessian matrix of $f \in C^2(\mathbb{R}^n)$ is positive semidefinite at a critical point x_0 of f, i.e.,

$$\langle v, Hf(x_0)v \rangle \geq 0 \text{ for all } v \in \mathbb{R}^n.$$

Which of the following statements necessarily hold? (There may be more than one).

(a) x_0 is a strict local minimum of f.

(b) x_0 is a local minimum of f.

(c) x_0 is not a local maximum of f.

(d) None of the above statements.

6.9. Minimization. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = 2x^2 + y^2 - x$. Determine the extrema of f on...

(a) ...the unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$;

(b) ...the closed unit disk $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$.

6.10. Lagrange Multipliers (♡). Consider the function $f(x, y, z) = 3x - y + 2z$ and the set

$$M = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1, x + y = 0\}.$$

Determine the extrema of f on M and their nature.

6.11. Critical Points. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = (ax^2 + by^2)e^{-x^2-y^2}$ with real parameters $a, b \in \mathbb{R}$. Find all critical points and determine their nature with the Hessian test, depending on a, b.

Hints:

6.2 Use the spectral Theorem and the properties:

$$\det(AB) = \det(A)\det(B), \quad \mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

6.3 Minimize A^2 instead of A. You can use the method of Lagrange multipliers.

6.5 Do not get distracted by the setting, you after all you have to minimize a quadratic polynomial of a, b....