Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems/subquestions marked with (♡).

7.1. BONUS PROBLEM.

(a) Give a diffeomorphism between $\mathbb{R}^2$ and $(0,1) \times (0,1)$.

(b) Is $f(x) = x^5, x \in \mathbb{R}$ a diffeomorphism of $\mathbb{R}$ to itself? Motivate rigorously your answer.

7.2. Inverse function I. (♡) Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = (x^2y, xy^2)$. Show that F is locally invertible around all points (x,y) such that $x \neq 0$ and $y \neq 0$. Compute the differential of the local inverse of F at the point $F(2,1)$.

7.3. Implicit Function I. (♡) Sketch the zero set of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$:

1. $f(x,y) = x^2 + y^2 - 1$,
2. $f(x,y) = y^2(1-x) - x^3$,
3. $f(x,y) = y^2 - x^2(x+1)$,
4. $f(x,y) = xy(x+y-1)$,
5. $f(x,y) = x^2y^2 - x^2 - y^2 + 1$.

You can also use software to help you.

At which points $(x_0, y_0) \in \mathbb{R}^2$, does the Implicit Function Theorem imply that the function can be locally resolved with respect to x (or with respect to y, with respect to both, or possibly with respect to neither variable)? Mark these points in your sketch.

7.4. Multiple choice. (♡) Mark all and only the true statements

(a) Let $U \subset \mathbb{R}^n$ be open and $f \in C^1(U, \mathbb{R}^n)$ such that $\det Jf(x) > 0$ for all $x \in U$. Then the set $f(U)$ is open.

(b) Let $U \subset \mathbb{R}^n$ be open and $f \in C^1(U, \mathbb{R}^n)$ such that $\det Jf(x) > 0$ for all $x \in U$. Then f is injective.

(c) Is there a diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(U) = V$, with $U := \{x^2 + y^2 < 1\} \subset \mathbb{R}^2$ and $V := \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$.

(d) (*) Let T be a triangle and Q be a square on the plane (just the boundary, not the interior). Is there a diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(T) = Q$?

7.5. Implicit function II. Show that the system of equations

$$\begin{cases} xy^5 + yu^5 + zv^5 = 1, \\ x^5y + y^5u + z^5v = 1, \end{cases}$$

is solvable for the variables u and v in a neighborhood of the point $(x_0, y_0, z_0, u_0, v_0) = (0, 1, 1, 1, 0)$ and determine the derivatives $D_{(0,1,1)}u$ and $D_{(0,1,1)}v$ of the implicitly defined functions $u = u(x, y, z)$ and $v = v(x, y, z)$.

7.6. Spherical Coordinates. The mapping $\Phi\colon (0, \infty) \times (0, \pi) \times (-\pi, \pi) \to \mathbb{R}^3$ defined by

$$\Phi(r, \theta, \varphi) = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

is called *spherical coordinates.*

1. Sketch the images of $r \mapsto \Phi(r, \theta_0, \varphi_0)$, $\theta \mapsto \Phi(r_0, \theta, \varphi_0)$ and $\varphi \mapsto \Phi(r_0, \theta_0, \varphi)$ for some fixed $r_0 \in (0, \infty)$, $\theta_0 \in (0, \pi)$, $\varphi_0 \in (-\pi, \pi)$.
2. What is the image of Φ?
3. Show that $\det(D_{(r,\theta,\varphi)}\Phi) = r^2 \sin\theta$ holds.
4. Conclude that the mapping Φ is a diffeomorphism onto its image.

7.7. The IFT is only a sufficient condition. We consider the function $f(x, y) = y^2(1 - x) - x^3$ from Exercise 6.3.3 in more detail.

1. Show that we cannot conclude from the implicit function theorem that f is solvable for x in a neighborhood of $(0, 0)$.
2. Show, however, that the equation $f(x, y) = 0$ can be uniquely solved for x everywhere. Hint: Analyze the mapping $x \mapsto \frac{x^3}{1-x}$ on a suitable domain.
3. Denote by $Y(x) > 0$ the function such that $f(x, Y(x)) = 0$ around $x = 1/2$. Compute $Y''(1/2)$. Hint: Derive twice with respect to x the identity $f(x, Y(x)) = 0$ end evaluate it at $x = 1/2$.

7. Solutions

Solution of 7.1:

(a) $(x, y) \mapsto (\frac{1}{2}, \frac{1}{2}) + \frac{1}{\pi}(\arctan(x), \arctan(y))$

(b) No, because $f'(0) = 0$. More extensively: if f was a diffeomorphism then let g denote the inverse. It would hold $g'(f(x)) = 1/f'(x) = x^{-5}/5$ which is impossible as $x \to 0$. On the other hand g should be regular in a neighbourhood of $f(0)$.

Solution of 7.2: The jacobi matrix of f is

$$JF(x, y) = \begin{bmatrix} 2xy & x^2 \\ y^2 & 2xy \end{bmatrix}, \quad \det JF(x, y) = 3x^2y^2,$$

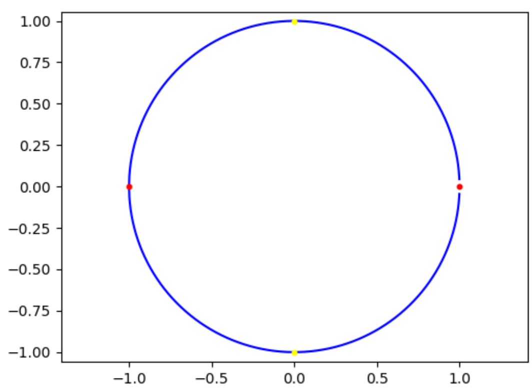
which vanish exactly in the set $\{x = 0\} \cup \{y = 0\}$. By the formula

$$JF^{-1}(4, 2) = JF^{-1}(F(2, 1)) = JF(2, 1)^{-1} = \begin{bmatrix} 4 & 4 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{12}\begin{bmatrix} 4 & -4 \\ -1 & 4 \end{bmatrix}.$$

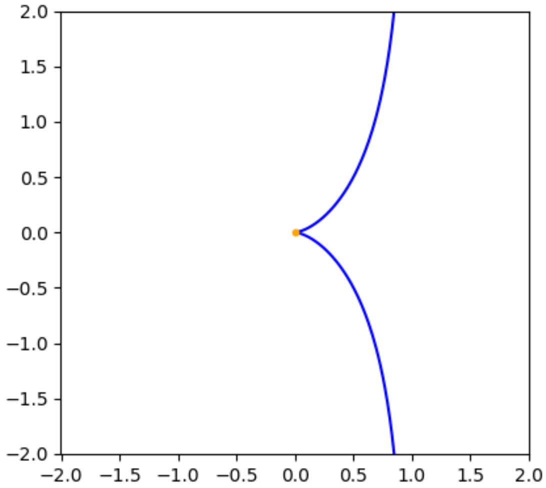
Solution of 7.3: Below, we draw the set of roots $N = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ of the function f. Using the implicit function theorem, we differentiate the following cases:

- the points where one can locally resolve with respect to x and y, in blue (i.e., $\{(x, y) \in N \mid \partial_x f(x, y), \partial_y f(x, y) \neq 0\}$),
- the points where one can locally resolve with respect to y but possibly not with respect to x, in yellow (i.e., $\{(x, y) \in N \mid \partial_x f(x, y) = 0, \partial_y f(x, y) \neq 0\}$),
- the points where one can locally resolve with respect to x but possibly not with respect to y, in red (i.e., $\{(x, y) \in N \mid \partial_x f(x, y) \neq 0, \partial_y f(x, y) = 0\}$),
- the points where one can locally possibly resolve neither with respect to x nor with respect to y, in orange (i.e., $\{(x, y) \in N \mid \partial_x f(x, y) = \partial_y f(x, y) = 0\}$).

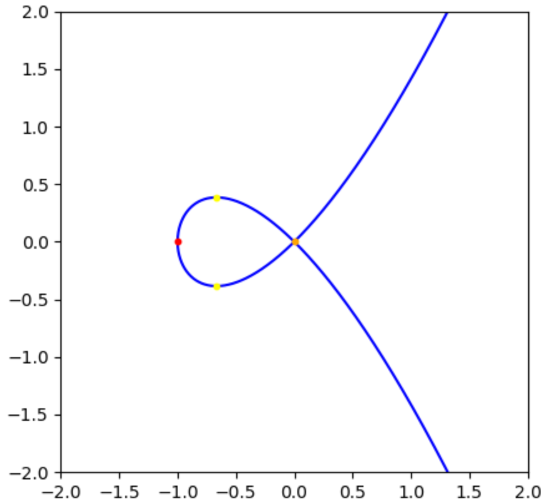
1. $D_{(x,y)} f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$



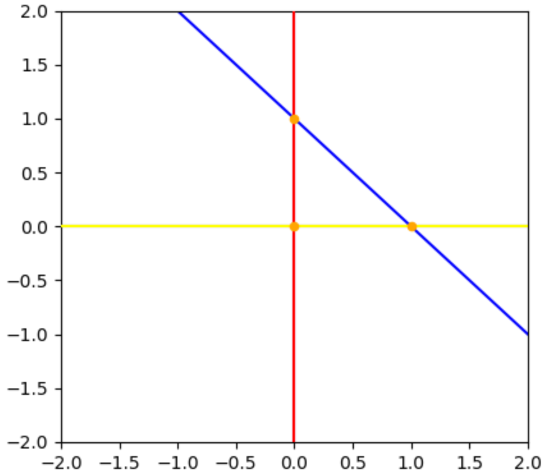
2. $D_{(x,y)}f = \begin{pmatrix} -(y^2+3x^2) \\ 2y(1-x) \end{pmatrix}$



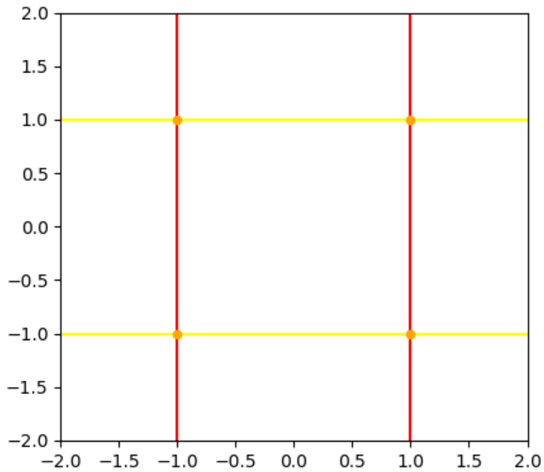
3. $D_{(x,y)}f = \begin{pmatrix} -x(3x+2) \\ 2y \end{pmatrix}$



4. $D_{(x,y)}f = \begin{pmatrix} y(2x+y-1) \\ x(x+2y-1) \end{pmatrix}$



5. The function is $f(x,y) = (x^2-1)(y^2-1)$. We get $D_{(x,y)}f = \begin{pmatrix} 2x(y^2-1) \\ 2y(x^2-1) \end{pmatrix}$



Solution of 7.4:

(a) True, this is part of the statement of the Inverse function theorem.

(b) False, globally F might fail to be injective. Take for example the complex exponential given in polar coordinates by $(r,\theta) \mapsto (e^r \sin(\theta), e^r \cos(\theta))$, with $x = r\cos\theta, y = r\sin\theta$.

(c) No. Since V is compact and ϕ^{-1} is continuous, then $\phi^{-1}(V) = U$ must be compact and thus closed as subset of $\mathbb{R}^n$. But U is also open in $\mathbb{R}^n$, so U is clopen in $\mathbb{R}^n$ which is connected, contradiction.

(d) No, the number of angular points must be preserved and $3 \neq 4$.
To prove the fact that the number of angular points must be preserved by a diffeomorphism, we can model the triangle T as the image of a closed path γ, i.e. a continuous map $\gamma\colon [0,1] \to \mathbb{R}^2$, such that $\gamma(0) = \gamma(1)$. This map γ is also differentiable except at the three points $A, B, C \in [0,1]$, where the derivative $\gamma'(x)$ suffers from a jump-discontinuity. We can arrange the choice of γ in such a way that $\gamma'(x) \neq 0$ at all differentiability points. The points $\gamma(A), \gamma(B), \gamma(C)$ correspond precisely to the three vertices of the triangle T in $\mathbb{R}^2$, and they are the angular points of the figure.
Assume now, by contradiction, that $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism such that $\phi(T) = Q$, then we can express the square Q as the image of the closed path $\phi \circ \gamma$. We can then calculate the derivative $(\phi \circ \gamma)'(x)$ using the chain rule:

$$(\phi \circ \gamma)'(x) = D\phi(\gamma(x)) \cdot \gamma'(x) \qquad \Longleftrightarrow \qquad \gamma'(x) = [D\phi(\gamma(x))]^{-1} \cdot (\phi \circ \gamma)'(x)$$

By the invertibility of $D\phi$ due to the diffeomorphism property, we have explicitly constructed a $1-1$ correspondence between points of differentiability of γ and points of differentiability of $\phi \circ \gamma$, hence we also have a $1-1$ correspondence between the respective points of non-differentiability of the two curves.
Hence there must be an $\bar{x} \in [0,1]$ such that γ is differentiable at $\bar{x}$, $\gamma'(\bar{x}) \neq 0$ and $\phi(\gamma(\bar{x}))$ is one of the corners of Q (say the top right one). The contradiction is that the functions

$$e_1 \cdot (\phi \circ \gamma)(x), \quad e_2 \cdot (\phi \circ \gamma)(x)$$

have both a local maximum at $\bar{x}$, so $(\phi \circ \gamma)'(\bar{x}) = 0$, so the contradiction

$$0 \neq \gamma'(\bar{x}) = [D\phi(\gamma(\bar{x}))]^{-1} \cdot \underbrace{(\phi \circ \gamma)'(\bar{x})}_{=0} = 0.$$

Solution of 7.5: We write the given system of equations as $F(x,y,z,u,v) = 0$ for the function $F\colon \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$F(x,y,z,u,v) = \begin{pmatrix} xy^5 + yu^5 + zv^5 - 1 \\ x^5y + y^5u + z^5v - 1 \end{pmatrix}.$$

The derivative is

$$D_{(x,y,z,v,u)}F = \begin{pmatrix} y^5 & 4xy^4 + u^5 & v^5 & 5yu^4 & 5zv^4 \\ 5x^4y & x^5 + 5y^4u & 5z^4v & y^5 & z^5 \end{pmatrix}$$

and thus

$$D_{(0,1,1,1,0)}F = \begin{pmatrix} 1 & 1 & 0 & 5 & 0 \\ 0 & 5 & 0 & 1 & 1 \end{pmatrix}.$$

According to the implicit function theorem (Theorem 11.1, 11.2) applied to $(0,1,1,1,0)$, we need to check that the submatrix of the differential DF consisting of the partial derivatives with respect to u and v is invertible. Indeed, the relevant submatrix is the matrix $\left(\begin{smallmatrix} 5 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ with determinant $5 \neq 0$. Therefore, by the implicit function theorem, for the locally defined function $G(x, y, z) = (u(x, y, z), v(x, y, z))$, we have

$$D_{(0,1,1)}G = -\begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 5 & 0 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 24 & 0 \end{pmatrix}.$$

So, $D_{(0,1,1)}u = -\frac{1}{5}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ and $D_{(0,1,1)}v = -\frac{1}{5}\begin{pmatrix} -1 & 24 & 0 \end{pmatrix}$.

Solution of 7.6:

1.
   - Fixing $\theta_0 \in (0, \pi)$, $\varphi_0 \in (-\pi, \pi)$, $p_0 = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$ defines a point on the sphere. The image of $r \mapsto \Phi(r, \theta_0, \varphi_0)$ consists of the points rp_0 with $r \in (0, \infty)$, forming a radial line.
   - Fixing $r_0 \in (0, \infty)$, $\varphi_0 \in (-\pi, \pi)$, the image of $\theta \mapsto \Phi(r_0, \theta, \varphi_0)$ is a semicircle on the sphere with radius r_0, ranging from the north pole to the south pole at longitude φ_0 (excluding the poles). Negative φ_0 is sometimes denoted as *West*, and positive φ_0 as *East*. In geography, degrees are often used instead of radians for angles.
     (see `https://en.wikipedia.org/wiki/Meridian_(geography)`)
   - Fixing $r_0 \in (0, \infty)$, $\theta_0 \in (0, \pi)$, the image of $\varphi \mapsto \Phi(r_0, \theta_0, \varphi)$ is a parallel circle on the sphere with radius r_0 at latitude θ_0. In geography, latitude is measured as a value in degrees in $(90°\mathrm{S}, 90°\mathrm{N})$, which corresponds to the angle $\frac{\pi}{2} - \theta_0$ in mathematics. Note that the point on the equator is not included in the image.
     (see `https://en.wikipedia.org/wiki/Circle_of_latitude`)
2. For each r, the image is the sphere of radius r excluding the prime meridian. Overall, the image is

$$\mathbb{R}^3 \setminus \{(x, 0, z) \in \mathbb{R}^3 \mid x \leq 0\} = \mathbb{R}^3 \setminus (-\infty, 0] \times \{0\} \times \mathbb{R}).$$

3. We compute:

$$D_{(r,\theta,\varphi)} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

hence

$$\begin{aligned} D_{(r,\theta,\varphi)} &= r^2\sin\theta\cos^2\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi + r^2\sin\theta\cos^2\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi \\ &= r^2(\sin\theta\cos^2\theta + \sin^3\theta) = r^2\sin\theta. \end{aligned}$$

4. Since $r \in (0, \infty)$ and $\theta \in (0, \pi)$, $r^2 \sin\theta$ is never 0. Thus, Φ is a local diffeomorphism. To show that Φ is a global diffeomorphism onto its image, we need to show that Φ is injective:
   If $\Phi(r_1, \theta_1, \varphi_1) = \Phi(r_2, \theta_2, \varphi_2)$, then $r_1 = r_2$ follows when taking the magnitude. Since $\cos : (0, \pi) \to (-1, 1)$ is injective, we have $\theta_1 = \theta_2$ from the third coordinate. Finally, we find $\varphi_1 = \varphi_2$, since $(\cos\varphi, \sin\varphi)$ with $\varphi \in (-\pi, \pi)$ describes a unique point on the unit circle.

Solution of 7.7:

1. We have $D_{(0,0)}f = \begin{pmatrix} 0 & 0 \end{pmatrix}$, thus $\partial_y f(0,0) = 0$. Therefore, the theorem is not applicable at $(0,0)$.

2. Let $y \in \mathbb{R}$. We want to show that there exists a unique $x \in \mathbb{R}$ such that $y^2(1-x) - x^3 = 0$. Note that when $x = 1$, the equation $y^2(1-x) - x^3 = 0$ cannot be satisfied. Moreover, from $y^2 = \frac{x^3}{1-x}$, it follows that the right-hand side must be non-negative, as it is a square of a number. Let $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be the function

$$x \mapsto \frac{x^3}{1-x}$$

   The pre-image of non-negative values $y^2 \in [0, \infty)$ is $x \in [0, 1)$. We need to show that the mapping is unique, i.e., g is injective on $[0, 1)$. The derivative of g is

$$g'(x) = \frac{(3-2x)x^2}{(1-x)^2},$$

   thus for $x \in [0, 1)$, we always have $g'(x) > 0$, with equality only when $x = 0$. Together with the fundamental theorem of calculus, this implies that g is strictly increasing on $[0, 1)$. Since $f(0) = 0$ and $\lim_{x \to 1} f(x) = \infty$, $f : [0, 1) \to [0, \infty)$ is bijective.

3. First of all if $x = 1/2$ then

$$0 = \tfrac{1}{2}Y(\tfrac{1}{2})^2 - \tfrac{1}{8}, \quad Y(\tfrac{1}{2}) > 0 \iff Y(\tfrac{1}{2}) = \tfrac{1}{2}.$$

   Differentiating we find

$$0 = 2Y(x)Y'(x)(1-x) - Y(x)^2 - 3x^2 \iff 0 = 2x^3Y'(x) - Y(x)^3 - 3x^2Y(x).$$

   Which gives, evaluating at $x = \frac{1}{2}$ and using $Y(\frac{1}{2}) = \frac{1}{2}$,

$$\tfrac{1}{4}Y'(\tfrac{1}{2}) = \tfrac{1}{8} + \tfrac{3}{8} \iff Y'(\tfrac{1}{2}) = 1.$$

   Differentiating once again

$$6x^2Y'(x) + 2x^3Y''(x) = 3Y(x)^2Y'(x) + 6xY(x) + 3x^2Y'(x)$$

   Which gives, evaluating at $x = \frac{1}{2}$, and using $Y(\frac{1}{2}) = \frac{1}{2}, Y'(\frac{1}{2}) = 1$ that

$$\tfrac{3}{2} + \tfrac{1}{4}Y''(\tfrac{1}{2}) = \tfrac{3}{4} + \tfrac{3}{2} + \tfrac{3}{4} \iff Y''(\tfrac{1}{2}) = 6.$$