Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems/subquestions marked with (♡).

8.1. BONUS PROBLEM. Let $X \subset \mathbb{R}$ be a Jordan-null set (as in Definition 13.8).

(a) Show rigorously that $X \times X \subset \mathbb{R}^2$ is also Jordan-null.

(b) Show rigorously that $X \times [0,1] \subset \mathbb{R}^2$ is also Jordan-null.

8.2. True or False. (♡)

1. A bounded countable set is always Jordan-null.
2. A countable set is always Lebesgue-null.
3. Let $D \subset [0,1]$ be a dense set (i.e., $\overline{D} = [0,1]$). Then $\mu_{out}(D) = 1$. (μ_{out} was defined in Definition 13.7).
4. Let $X, Y \subset [0,1]$ Jordan measurable sets such that $\mu(X) > 1/2$ and $\mu(Y) > 1/2$. Then $X \cap Y \neq \emptyset$.
5. Let $X, Y \subset [0,1]$ such that $\mu_{out}(X) > 1/2$ and $\mu_{out}(Y) > 1/2$. Then $X \cap Y \neq \emptyset$.

8.3. Fat Boundary. Construct an open subset $U \subset \mathbb{R}$ for which the boundary ∂U is not a null set.

8.4. Multiple Choice. (♡) Let $U \subset \mathbb{R}^n$ be a nonempty, open subset, $f\colon U \to \mathbb{R}^m$ a function, and $N \subset U$ a Jordan null set. In which of the following cases is the image $f(N) \subset \mathbb{R}^m$ necessarily a null set? Attention: Only one answer is correct!

1. If f is uniformly continuous.
2. If f is uniformly continuous and $m \geq n$.
3. If f is locally Lipschitz continuous.
4. If f is locally Lipschitz continuous and $m \geq n$.

8.5. Change of variables and Jacobians. (♡) For each of the following domains and change of variables find the jacobian and the appropriate transformed domain. There is no need to actually compute the integrals!

1. $A := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 < x_1, 1 < x_1^2 + x_2^2 < 4\}$ and $x_1 = r\cos\theta, x_2 = r\sin\theta$. Using the change of variables formula, complete the dots in the following formula

$$\int_A x_1^2 \sin(x_2)\, dx_1 dx_2 = \int_{...} ...\, dr d\theta$$

2. $B := \{(x,y) \in \mathbb{R}^2 \,|\, 1 < xy < 2, x^2 < y < 2x^2\}$ and $u := xy, v := x^2$. Using the change of variables formula, complete the dots in the following formula

$$\int_B y^2 e^{-xy} dxdy = \int_{...} ...\, dudv$$

3. $C := \{(x,y,z) \in \mathbb{R}^3 \,|\, 1 < z - 2y < 2, 0 < z < 1, -2 < 3x + y + z < 0\}$ and $u := z, v := z - 2y, w := 3x + y + z$. Using the change of variables formula, complete the dots in the following formula

$$\int_C xyz \; dxdy = \int_{...} ...\, dudvdw.$$

8.6. The Cantor set. (*) Let $X \subset [0,1]$ be the set of all real numbers whose decimal expansion does not contain the digit 8.[1] Show that:

1. X is a Lebesgue null set,
2. X is uncountable,
3. $X \times X \subset [0,1]^2$ is a Lebesgue null set.
4. Show that X is compact (it is important the choice made in the footnote!).

[1] The decimal expansion is not always unique. For example, $0.8 = 0.79999\ldots$. Whenever x has at least one decimal expansion not containing 8, we rule that x **belongs to** X, so for example $0.3257\overline{9} \in X$, $0.3258\overline{9} \in X$

Hints:

8.2.5 μ_{out} can be positive and large on very sparse sets...

8.3 Any open subset of $\mathbb{R}$ is an union of disjoint open intervals. Try to achieve that U has a very small "total volume", but still contains all rational numbers in $[0, 1]$.