Problems marked with a (*) are a bit more complex and can be skipped at a first read.

If you don't have a lot of time focus on the Problems/subquestions marked with (♡).

8.1. BONUS PROBLEM. Let $X \subset \mathbb{R}$ be a Jordan-null set (as in Definition 13.8).

(a) Show rigorously that $X \times X \subset \mathbb{R}^2$ is also Jordan-null.

(b) Show rigorously that $X \times [0,1] \subset \mathbb{R}^2$ is also Jordan-null.

8.2. True or False. (♡)

1. A bounded countable set is always Jordan-null.
2. A countable set is always Lebesgue-null.
3. Let $D \subset [0,1]$ be a dense set (i.e., $\overline{D} = [0,1]$). Then $\mu_{out}(D) = 1$. (μ_{out} was defined in Definition 13.7).
4. Let $X, Y \subset [0,1]$ Jordan measurable sets such that $\mu(X) > 1/2$ and $\mu(Y) > 1/2$. Then $X \cap Y \neq \emptyset$.
5. Let $X, Y \subset [0,1]$ such that $\mu_{out}(X) > 1/2$ and $\mu_{out}(Y) > 1/2$. Then $X \cap Y \neq \emptyset$.

8.3. Fat Boundary. Construct an open subset $U \subset \mathbb{R}$ for which the boundary ∂U is not a null set.

8.4. Multiple Choice. (♡) Let $U \subset \mathbb{R}^n$ be a nonempty, open subset, $f\colon U \to \mathbb{R}^m$ a function, and $N \subset U$ a Jordan null set. In which of the following cases is the image $f(N) \subset \mathbb{R}^m$ necessarily a null set? Attention: Only one answer is correct!

1. If f is uniformly continuous.
2. If f is uniformly continuous and $m \geq n$.
3. If f is locally Lipschitz continuous.
4. If f is locally Lipschitz continuous and $m \geq n$.

8.5. Change of variables and Jacobians. (♡) For each of the following domains and change of variables find the jacobian and the appropriate transformed domain. There is no need to actually compute the integrals!

1. $A := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 < x_1, 1 < x_1^2 + x_2^2 < 4\}$ and $x_1 = r\cos\theta, x_2 = r\sin\theta$. Using the change of variables formula, complete the dots in the following formula

$$\int_A x_1^2 \sin(x_2)\, dx_1 dx_2 = \int_{...} ...\, dr d\theta$$

2. $B := \{(x,y) \in \mathbb{R}^2 \,|\, 1 < xy < 2, x^2 < y < 2x^2\}$ and $u := xy, v := x^2$. Using the change of variables formula, complete the dots in the following formula
$$\int_B y^2 e^{-xy} dxdy = \int_{...} ... \, dudv$$

3. $C := \{(x,y,z) \in \mathbb{R}^3 \,|\, 1 < z - 2y < 2, 0 < z < 1, -2 < 3x + y + z < 0\}$ and $u := z, v := z - 2y, w := 3x + y + z$. Using the change of variables formula, complete the dots in the following formula
$$\int_C xyz \; dxdy = \int_{...} ... \, dudvdw.$$

8.6. The Cantor set. (*) Let $X \subset [0,1]$ be the set of all real numbers whose decimal expansion does not contain the digit 8.[1] Show that:

1. X is a Lebesgue null set,
2. X is uncountable,
3. $X \times X \subset [0,1]^2$ is a Lebesgue null set.
4. Show that X is compact (it is important the choice made in the footnote!).

[1] The decimal expansion is not always unique. For example, $0.8 = 0.79999\ldots$. Whenever x has at least one decimal expansion not containing 8, we rule that x **belongs to** X, so for example $0.3257\overline{9} \in X$, $0.3258\overline{9} \in X$

Hints:

8.2.5 μ_{out} can be positive and large on very sparse sets...

8.3 Any open subset of $\mathbb{R}$ is an union of disjoint open intervals. Try to achieve that U has a very small "total volume", but still contains all rational numbers in $[0, 1]$.

8. Solutions

Solution of 8.1: For $\epsilon > 0$ there is a finite collection of dyadic intervals $\{I_1, \ldots, I_N\}$ such that

$$X \subset I_1 \cup \ldots \cup I_N, \quad \sum_{1 \le 1 \le N} \mu_1(I_i) \le \epsilon.$$

Now we must have

$$X \times [0,1] \subset I_1 \times [0,1] \cup \ldots \cup I_N \times [0,1],$$

and so

$$\mu_{out}(X \times [0,1]) \le \sum_{1 \le i \le N} \mu_2(I_i \times [0,1]) = \sum_{1 \le 1 \le N} \mu_1(I_i) \le \epsilon.$$

Since ϵ was arbitrary, $\mu_{out}(X \times [0,1]) = 0$. Since without loss of generality we may have assumed that $X \subset [0,1]$ then

$$\mu_{out}(X \times X) \le \mu_{out}(X \times [0,1]) = 0.$$

Solution of 8.2:

1. False, $\mathbb{Q}$ is acounterexample.
2. True, number it's elements $\{a_k\}_{k \in \mathbb{N}}$ and cover it with $\{a_k + \epsilon 2^{-k}[-1,1]\}_{k \in \mathbb{N}}$.
3. True, pick any finite dyadic partition of $[0,1]$. Then each interval must contain at least one element of D by density. Hence if we want to cover D we must keep all these intervals, whose length sums to 1.
4. True. By contradiction, assume that $X \cap Y = \emptyset$. By Theorem 13.18, we must have $\mu(X \cup Y) = \mu(X) + \mu(Y) > 1$. On the other hand, since $X \cup Y \subseteq [0,1]$, it holds $\mu(X \cup Y) \le \mu([0,1]) = 1$, which gives the desired contradiction.
5. False, take $X = \mathbb{Q} \cap [0,1]$ and $Y := (\sqrt{2} + \mathbb{Q}) \cap [0,1]$.

Solution of 8.3: Let $0 < \epsilon < 1/2$ and $(q_n)_{n \in \mathbb{N}}$ be a enumeration of $[0,1] \cap \mathbb{Q}$. We define $I_n := (q_n - \epsilon 2^{-n}, q_n + \epsilon 2^{-n})$ and

$$U := \bigcup_{n \in \mathbb{N}} I_n.$$

By construction, we then have $[0,1] \cap \mathbb{Q} \subset U$ and the "total volume"[2] is small, in the sense that

$$\sum_{n=1}^{\infty} \mathrm{vol}(I_n) = 2\epsilon \sum_{n=1}^{\infty} 2^{-n} = 2\epsilon. \tag{1}$$

We now show that this set U satisfies the requirements in the problem statement. Being a union of open intervals, U is certainly open in $\mathbb{R}$. Next, we notice that due to $[0,1] \cap \mathbb{Q} \subset U$,

[2] We always write this in quotation marks because the point of this exercise is that this set U is not Jordan-measurable.

the entire interval $[0,1]$ is contained in the closure $\overline{U}$ of U. Using the boundary of U, this can be written as

$$\partial U \cup U = (\overline{U} \setminus U) \cup U = \overline{U} \supset [0,1].$$

In view of (1), it is therefore natural to conjecture that for a covering of ∂U by open boxes $(Q_l)_l$ in $\mathbb{R}$ (that is, by intervals), it is necessary that

$$\sum_{l=1}^{\infty} \mathrm{vol}(Q_l) \geq 1 - 2\epsilon \tag{2}$$

must hold. To prove this lower bound, we first notice that for such a covering of the boundary

$$[0,1] \subset \partial U \cup U \subset \bigcup_{n\in\mathbb{N}} I_n \cup \bigcup_{l\in\mathbb{N}} Q_l$$

is an open covering of the compact interval $[0,1]$. Therefore, finitely many of these sets suffice to cover it; thus, there exist $N, L \in \mathbb{N}$ such that

$$[0,1] \subset \bigcup_{n=1}^{N} I_n \cup \bigcup_{l=1}^{L} Q_l.$$

Both the I_n and the Q_l are open intervals, whose volume is computed as the difference of the right and left endpoints. Elementary considerations[3] show that therefore for the volumes, it holds that

$$1 \leq \sum_{n=1}^{N} \mathrm{vol}(I_n) + \sum_{l=1}^{L} \mathrm{vol}(Q_l).$$

From this, using (1), we now obtain

$$\sum_{l=1}^{\infty} \mathrm{vol}(Q_l) \geq \sum_{l=1}^{L} \mathrm{vol}(Q_l) \geq 1 - \sum_{n=1}^{N} \mathrm{vol}(I_n) \geq 1 - \sum_{n=1}^{\infty} \mathrm{vol}(I_n) = 1 - 2\epsilon,$$

exactly as in (2). Since ϵ was chosen such that $1 - 2\epsilon > 0$, this shows that ∂U cannot be a null set.

Solution of 8.4: The only correct one is number 4 which is Lemma 13.6.

Solution of 8.5:

[3] If an interval $[a,b]$ is covered by finitely many intervals $J_1, \ldots, J_r$, ordered increasingly by their left endpoints, then the right endpoint of J_k is at most $a + \sum_{j=1}^{k} \mathrm{vol}(J_j)$. Therefore, for the right endpoint b to also lie in one of the intervals J_j, it must hold that $\sum_{j=1}^{r} \mathrm{vol}(J_j) \geq b - a$.

1. If $x_1 = r\cos\theta, x_2 = r\sin\theta$ then we have

$$(x_1, x_2) \in A \iff (r, \theta) \in \{-\pi < \theta < \pi, \sin\theta < \cos\theta, 1 < r < 2\}$$

solving the inequality

$$-\pi < \theta < \pi, \sin\theta < \cos\theta \iff -\pi < \theta < \pi/4.$$

So, recalling that the jacobian is r, we find

$$\int_A x_1^2 \sin(x_2)\, dx_1 dx_2 = \int_1^2 \Big\{ \int_{-\pi}^{\pi/4} \cos^2\theta \sin(r\sin\theta) d\theta \Big\} r^3 dr.$$

2. Notice that $B \subset (0,\infty)^2$ so (x, y) are always positive and so are (u, v). We write the inverse mappings

$$x = \sqrt{v}, \quad y = \frac{u}{\sqrt{v}}.$$

So we compute the determinant of

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 0 & \frac{1}{2\sqrt{v}} \\ \frac{1}{\sqrt{v}} & * \end{bmatrix}$$

and so we find

$$dx\, dy = \frac{du\, dv}{2v}.$$

The set B changes into

$$\begin{aligned} (x,y) \in B &\iff (u,v) \in \{1 < u < 2, v\sqrt{v} < u < 2v\sqrt{v}\}, \\ &\iff (u,v) \in \{1 < u < 2, u^{2/3} < v < 2^{-2/3}u^{2/3}\}, \end{aligned}$$

and the integrand $y^2 e^{-xy}$ changes into

$$\frac{u^2}{v} e^u$$

so summing up we find

$$\int_B y^2 e^{-xy} dxdy = \int_1^2 u^2 \Big\{ \int_{u^{2/3}}^{2^{-2/3}u^{2/3}} v^{-2} e^v\, dv \Big\}\, du.$$

3. We compute the determinant

$$\det \frac{\partial(u,v,w)}{\partial(x,y,z)} = \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix} = 6,$$

so $dx\, dy\, dz = \frac{1}{6}\, du\, dv\, dw$. And the set becomes

$$(x,y,z) \in C \iff 0 < u < 1, \quad 1 < v < 2, \quad -2 < w < 0.$$

To change variables in the function we need to compute the inverse mapping:

$$x = \tfrac{1}{3}w + \tfrac{1}{6}v - \tfrac{1}{2}u, \quad y = \tfrac{1}{2}u - \tfrac{1}{2}v, \quad z = u.$$

So we find

$$xyz = -\tfrac{1}{6}uvw + \tfrac{1}{6}u^2w + \tfrac{1}{3}u^2v - \tfrac{1}{12}uv^2 - \tfrac{1}{4}u^3.$$

So putting everything together

$$\int_C xyz\,dx\,dy\,dz = v\frac{1}{72}\int_0^1\int_1^2\int_{-2}^0 (-2uvw + 2u^2w + 4u^2v - uv^2 - 3u^3)du\,dv\,dw.$$

Solution of 8.6:

1. Let A_n be the set that does not contain 8 up to the n-th decimal place. For example,

   $$A_1 = [0, 0.8] \cup [0.9, 1].$$

   The intervals that decompose A_1 have total volume 0.9. The second set has form

   $$A_2 = [0, 0.08] \cup [0.09, 0.18] \cup \cdots \cup [0.69, 0.78] \cup [0.79, 0.8] \cup [0.9, 0.98] \cup [0.99, 1].$$

   The intervals that decompose A_2 have total volume $9 \cdot 0.09 = 0.81$ (there are a total of 11 intervals: 7 have length 0.09; and 4 intervals are shorter, but one finds 2 pairs of 2 intervals each, which together also have length 0.09). We try to find a pattern and consider A_3. Treating the 4 shorter intervals in A_2 as if we had 2 intervals of length 0.09, we see that each interval in A_2 is divided into 9 intervals of length 0.009 (again by combining shorter intervals). Thus we have $\mathrm{vol}(A_3) = 0.009 \cdot 81 = 0.9^3$.

   Analogously, we find that A_n consists of finitely many intervals whose total volume

   $$\mathrm{vol}(A_n) = 0.9\mathrm{vol}(A_{n-1})$$

   which implies that $\mathrm{vol}(A_n) = 0.9^n$.

   To show that X is a Lebesgue null set, let $\epsilon > 0$. Choose $n \in \mathbb{N}$ large enough such that $0.9^n < \epsilon$. Since A_n consists of finitely many intervals covering X and the sum of the lengths of these intervals is $0.9^n < \epsilon$, the claim follows. As a complement for those who prefer explicit formulas: Let $(I_n^k)_{k=1}^{s_n}$ be the intervals defining A_n. Then we have

   $$X \subset A_n = \bigcup_{k=1}^{s_n} I_n^k \text{ and } \sum_{k=1}^{s_n} \mathrm{vol}(I_n) = 0.9^n < \epsilon.$$

   This shows that in fact X is also Jordan-null set.

2. If X were countable, we would write all numbers in a countably infinite list. As in the Cantor diagonal argument, we take a number defined by the diagonal and change each digit to a digit that is still not 8. This number is still in X, but does not coincide with any number in our list. Contradiction! So X is uncountable.

3. We can cover $X \times X$ by the boxes in $A_n \times [0,1]$. We have $\mathrm{vol}(A_n \times [0,1]) = \mathrm{vol}(A_n) = 0.9^n$ and the argument is analogous to part (a).

4. Each of the $\{A_i\}$ of step 1 is compact, being the union of finitely many closed disjoint intervals in $[0,1]$. Our set by definition is

$$X := \bigcup_{i \geq 1} A_i,$$

which is closed (being an intersection of closed sets) and bounded (since A_1 is bounded). Hence X is compact.