9.1. BONUS PROBLEM. Calculate the volume of the region $B \subset \mathbb{R}^3$ enclosed by the surfaces $x^2 + y^2 + z^2 = 8$ and $2z = x^2 + y^2$. Hint: use cylindrical coordinates.

9.2. Multiple Integrals.

1. Let $D = [0, 2] \times [0, 1]$. Calculate
$$\iint_D (x^3 + 3x^2y + y^3)\, dxdy.$$

2. Let $D \subset \mathbb{R}^2$ be the interior of the triangle with vertices $(0, 0)$, $(0, \pi)$, and (π, π). Calculate
$$\iint_D x \cos(x + y)\, dxdy.$$

3. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x > 1, y > 1, x + y < 3\}$. Calculate
$$\iint_D \frac{1}{(x + y)^3}\, dxdy.$$

9.3. Fubini Theorem. Compute the integrals
$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \sin(y^2)\ dydx, \qquad \int_{-1}^{1} \int_{|y|}^{1} (x + y)^2\ dxdy,$$

9.4. Counterexample to Fubini. Let $f \colon [0, \infty)^2 \to \mathbb{R}$, defined by
$$f(x) = \begin{cases} e^{y-x} & x > y \geq 0, \\ -e^{x-y}, & 0 \leq x \leq y, \end{cases}$$

Compute the iterated integrals:
$$\int_0^{\infty} \Big\{ \int_0^{\infty} f(x, y)\, dx \Big\}\, dy, \qquad \int_0^{\infty} \Big\{ \int_0^{\infty} f(x, y)\, dy \Big\}\, dx,$$

and show that they have different values. Explain why this does not contradict Fubini's Theorem.

9.5. Volume of the cone over a set. Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set, $n \geq 1$. Consider the "cone over Ω"
$$C\Omega := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq 1, x \in (1 - t)\Omega\}.$$

Using Fubini theorem and homogeneity of μ_n show that

$$\mu_{n+1}(C\Omega) = \frac{\mu_n(\Omega)}{n+1}.$$

Use this result to compute the n-volume of the n-simplex:

$$\mu_n(T_n) := \mu_n\Big(\{a_1 e_1 + \ldots + a_n e_n : 0 \le a_i \le 1, a_1 + \ldots + a_n \le 1\}\Big) = \frac{1}{n!},$$

where $e_1, \ldots, e_n$ denotes an orthonormal frame of $\mathbb{R}^n$. Hint: show $(n+1)T_{n+1} = T_n$.

9.6. Gaussian integrals. Let $n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$ be a symmetric positive definite matrix. Show that

$$\int_{\mathbb{R}^n} e^{-\langle Ax, x\rangle}\, dx = \frac{\pi^{n/2}}{\sqrt{\det(A)}},$$

Hint: start with the case where A is a diagonal matrix then use the Spectral theorem for the general case. You can use also that $\int_{\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$.

9.7. Layer-cake formula. (*) Let $f\colon \mathbb{R}^n \to [0, \infty)$ be a continuous function which vanish identically outside a compact set, and let $p \ge 1$. Using Fubini's Theorem show the Layer-cake formula

$$\int_{\mathbb{R}} f(x)^p\, dx = p \int_0^\infty t^{p-1} \mu_n(\{x \in \mathbb{R}^n : f(x) > t\})\, dt.$$

Find a similar formula for the integral of

$$\int_{\mathbb{R}} \Phi(f(x))\, dx = \int_0^\infty \quad \ldots \quad \mu_n(\{x \in \mathbb{R}^n : f(x) > t\})\, dt.$$

where $\Phi \in C^1(\mathbb{R})$ is any function such that $\Phi(0) = 0$. Hint: $f(x) = \int_0^{f(x)}\, dt$.