9.1. BONUS PROBLEM. Calculate the volume of the region $B \subset \mathbb{R}^3$ enclosed by the surfaces $x^2 + y^2 + z^2 = 8$ and $2z = x^2 + y^2$. Hint: use cylindrical coordinates.

9.2. Multiple Integrals.

1. Let $D = [0, 2] \times [0, 1]$. Calculate
$$\iint_D (x^3 + 3x^2y + y^3)\, dxdy.$$

2. Let $D \subset \mathbb{R}^2$ be the interior of the triangle with vertices $(0, 0)$, $(0, \pi)$, and (π, π). Calculate
$$\iint_D x \cos(x + y)\, dxdy.$$

3. Let $D = \{(x, y) \in \mathbb{R}^2 \,|\, x > 1, y > 1, x + y < 3\}$. Calculate
$$\iint_D \frac{1}{(x + y)^3}\, dxdy.$$

9.3. Fubini Theorem. Compute the integrals
$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \sin(y^2)\ dydx, \qquad \int_{-1}^{1} \int_{|y|}^{1} (x + y)^2\ dxdy,$$

9.4. Counterexample to Fubini. Let $f\colon [0, \infty)^2 \to \mathbb{R}$, defined by
$$f(x) = \begin{cases} e^{y-x} & x > y \geq 0, \\ -e^{x-y}, & 0 \leq x \leq y, \end{cases}$$

Compute the iterated integrals:
$$\int_0^{\infty} \Big\{ \int_0^{\infty} f(x, y)\, dx \Big\}\, dy, \qquad \int_0^{\infty} \Big\{ \int_0^{\infty} f(x, y)\, dy \Big\}\, dx,$$

and show that they have different values. Explain why this does not contradict Fubini's Theorem.

9.5. Volume of the cone over a set. Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set, $n \geq 1$. Consider the "cone over Ω"
$$C\Omega := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq 1, x \in (1 - t)\Omega\}.$$

Using Fubini theorem and homogeneity of μ_n show that

$$\mu_{n+1}(C\Omega) = \frac{\mu_n(\Omega)}{n+1}.$$

Use this result to compute the n-volume of the n-simplex:

$$\mu_n(T_n) := \mu_n\Big(\{a_1e_1 + \ldots + a_ne_n : 0 \le a_i \le 1, a_1 + \ldots + a_n \le 1\}\Big) = \frac{1}{n!},$$

where $e_1, \ldots, e_n$ denotes an orthonormal frame of $\mathbb{R}^n$. Hint: show $(n+1)T_{n+1} = T_n$.

9.6. Gaussian integrals. Let $n \in \mathbb{N}$ and $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$ be a symmetric positive definite matrix. Show that

$$\int_{\mathbb{R}^n} e^{-\langle Ax,x\rangle}\,dx = \frac{\pi^{n/2}}{\sqrt{\det(A)}},$$

Hint: start with the case where A is a diagonal matrix then use the Spectral theorem for the general case. You can use also that $\int_{\mathbb{R}} \exp(-x^2)\,dx = \sqrt{\pi}$.

9.7. Layer-cake formula. (*) Let $f\colon \mathbb{R}^n \to [0,\infty)$ be a continuous function which vanish identically outside a compact set, and let $p \ge 1$. Using Fubini's Theorem show the Layer-cake formula

$$\int_{\mathbb{R}} f(x)^p\,dx = p\int_0^\infty t^{p-1}\mu_n(\{x \in \mathbb{R}^n : f(x) > t\})\,dt.$$

Find a similar formula for the integral of

$$\int_{\mathbb{R}} \Phi(f(x))\,dx = \int_0^\infty \quad \ldots \quad \mu_n(\{x \in \mathbb{R}^n : f(x) > t\})\,dt.$$

where $\Phi \in C^1(\mathbb{R})$ is any function such that $\Phi(0) = 0$. Hint: $f(x) = \int_0^{f(x)}\,dt$.

9. Solutions

Solution of 9.1: Let $B := \{(x,y,z) :\ x^2+y^2+z^2 \le 8, x^2+y^2-2z \le 0\}$. We calculate

$$\mathrm{vol}(B) = \int_B d\mathrm{vol}$$

We use cylindrical coordinates. With $\rho = r^2$ (so that $d\rho = 2rdr$), it holds:

$$\mathrm{vol}(B) = \int_B r\ drdzd\phi = \frac{1}{2}\int_{\left\{\frac{1}{2}\rho<z<\sqrt{8-\rho}\right\}} d\rho dz d\phi = \int_0^{2\pi}\int_0^4 \sqrt{8-\rho} - \frac{1}{2}\rho\ d\rho d\phi = \frac{4}{3}(8\sqrt{2}-7)\pi$$

Solution of 9.2:

1. We use Fubini's theorem:

$$\begin{aligned}\int_D (x^3+3x^2y+y^3)\,\mathrm{dvol}(x,y) &= \int_0^2\int_0^1 (x^3+3x^2y+y^3)\,dy\,dx \\ &= \int_0^2 \left(x^3+\frac{3}{2}x^2+\frac{1}{4}\right)dx = \frac{17}{2}.\end{aligned}$$

2. We have $D = \{(x,y)\in\mathbb{R}^2 \,|\, 0<x<\pi, x<y<\pi\}$ and with Fubini's theorem we get:

$$\begin{aligned}\int_D x\cos(x+y)\,\mathrm{dvol}(x,y) &= \int_0^\pi\int_x^\pi x\cos(x+y)\,dy\,dx \\ &= \int_0^\pi [x\sin(x+y)]_{y=x}^{\pi}\ dx \\ &= -\int_0^\pi x(\sin x+\sin(2x))\,dx \\ &= \left[x(\cos x+\frac{1}{2}\cos(2x))\right]_0^\pi - \int_0^\pi (\cos x+\frac{1}{2}\cos(2x))\,dx \\ &= -\frac{\pi}{2},\end{aligned}$$

   where we used integration by parts.

3. We have $D = \{(x,y)\in\mathbb{R}^2 \,|\, 1<x<2, 1<y<3-x\}$ and with Fubini's theorem we get:

$$\begin{aligned}\int_D \frac{1}{(x+y)^3}\,\mathrm{dvol}(x,y) &= \int_1^2\int_1^{3-x}\frac{1}{(x+y)^3}\,dy\,dx \\ &= -\frac{1}{2}\int_1^2\left[\frac{1}{(x+y)^2}\right]_{y=1}^{3-x} dx \\ &= -\frac{1}{2}\int_1^2\left(\frac{1}{9}-\frac{1}{(x+1)^2}\right)dx \\ &= -\frac{1}{18}+\frac{1}{2}\int_2^3\frac{1}{x^2}\,dx = \frac{1}{36}\end{aligned}$$

Solution of 9.3: Since the primitive of $\sin(y^2)$ is not easy to compute, we use Fubini Theorem to swap the integrals (a similar trick was used in Example 13.42)

$$\begin{aligned}\int_0^{\sqrt{\pi}}\int_x^{\sqrt{\pi}} \sin(y^2)\ dydx &= \int_0^{\sqrt{\pi}}\int_0^{y} \sin(y^2)\ dxdy \\ &= \int_0^{\sqrt{\pi}} y\sin(y^2)\ dy = -\frac{1}{2}\cos(y^2)|_0^{\sqrt{\pi}} = 1.\end{aligned}$$

Similarly:

$$\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2\ dxdy = \int_0^1\int_{-x}^{x}(x+y)^2\ dydx = \frac{8}{3}\int_0^1 x^3\ dx = \frac{2}{3}.$$

Solution of 9.4: We compute

$$\begin{aligned}\int_0^\infty\int_0^\infty f(x,y)\ dxdy &= \int_0^\infty \left(\int_y^\infty e^{y-x}\ dx + \int_0^y -e^{x-y}\ dx\right)\ dy = \\ &= \int_0^\infty \left(1 - 1 + e^{-y}\right)\ dy = \int_0^\infty e^{-y}\ dy = 1\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty\int_0^\infty f(x,y)\ dydx &= \int_0^\infty \left(\int_0^x e^{y-x}\ dx + \int_x^\infty -e^{x-y}\ dy\right)\ dx = \\ &= \int_0^\infty \left(1 - e^{-x} - 1\right)\ dx = \int_0^\infty -e^{-x}\ dx = -1\end{aligned}$$

Solution of 9.5: Since $\Omega \subset [-C,C]^n$ we have that also $C\Omega$ is bounded:

$$(x,t) \in C\Omega \implies 0 \le t \le 1, x \in [-C,C]^n.$$

We write our set using indicator functions, try to write it as a product

$$\mathbf{1}_{C\Omega}(x,t) = \mathbf{1}_{[0,1]}(t)\cdot \mathbf{1}_{(1-t)\Omega}(x) = \mathbf{1}_{[0,1]}(t)\cdot\mathbf{1}_{\Omega}(x/(1-t)),$$

we use this factorisation to apply Fubini and the change of variables $x = (1-t)y$,

$$\begin{aligned}\mu_{n+1}(C\Omega) &= \int_{\mathbb{R}^{n+1}} \mathbf{1}_{C\Omega}\, dxdt = \int_{\mathbb{R}}\Big\{\int_{\mathbb{R}^n}\mathbf{1}_{[0,1]}(t)\cdot\mathbf{1}_{\Omega}(x/(1-t))dx\Big\}dt \\ &= \int_{\mathbb{R}}\mathbf{1}_{[0,1]}(t)(1-t)^n\Big\{\int_{\mathbb{R}^n}\mathbf{1}_{\Omega}(y)dy\Big\}dt \\ &= \mu_n(\Omega)\int_0^1(1-t)^n dt = \mu_n(\Omega)\int_0^1 s^n ds = \frac{\mu_n(\Omega)}{n+1}.\end{aligned}$$

Clearly $T_1 = [0,1] \subset \mathbb{R}$ and $\mu_1(T_1) = 1$. We claim that

$$T_{n+1} = CT_n,$$

from which the formula follows recursively:

$$\mu_{n+1}(T_{n+1}) = \mu_{n+1}(CT_n) = \frac{\mu_n(T_n)}{n+1} = \frac{\mu_n(CT_{n-1})}{n+1} = \frac{\mu_{n-1}(T_{n-1})}{(n+1)n} = \ldots = \frac{\mu_1(T_1)}{(n+1)!}.$$

We check the claim: Let $0 \leq a_i \leq 1$ for $i = 1, \ldots, n+1$ and $x = a_1 e_1 + \ldots + a_{n+1} e_{n+1}$, then

$$\begin{aligned} x \in T_{n+1} &\iff a_1 + \ldots + a_{n+1} \leq 1 \iff \frac{a_1 + \ldots + a_n}{1 - a_{n+1}} \leq 1 \\ &\iff \frac{a_1}{1-a_{n+1}} e_1 + \ldots + \frac{a_n}{1-a_{n+1}} e_n \in T_n \\ &\iff (a_1 e_1 + \ldots + a_n e_n, a_{n+1}) \in CT_n \iff x \in CT_n. \end{aligned}$$

Solution of 9.6: According to the spectral Theorem, we can write A as $A = KDK^{-1}$ for an orthogonal matrix $K \in \mathrm{O}_n(\mathbb{R})$ and a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathrm{Mat}_{n,n}(\mathbb{R})$$

with $\lambda_i > 0$ for $1 \leq i \leq n$. Due to $K^t = K^{-1}$ and $|\det(K)| = 1$, it follows from the substitution rule

$$\int_{\mathbb{R}^n} e^{-\langle Ax,x\rangle} \mathrm{dvol}(x) = \int_{\mathbb{R}^n} e^{-\langle DK^{-1}x, K^{-1}x\rangle} \mathrm{dvol}(x) = \int_{\mathbb{R}^n} e^{-\langle Dx,x\rangle} \mathrm{dvol}(x), \tag{1}$$

provided one of these improper integrals exists. Let $(B_m)_m$ be the exhaustion of $\mathbb{R}^n$ consisting of the closed cubes $B_m = [-m, m]^n \subset \mathbb{R}^n$. Then, $x \mapsto e^{-\langle Dx,x\rangle}$ is Riemann integrable as a continuous function over each of the sets B_m, and by Fubini's theorem, we have

$$\int_{B_m} e^{-\langle Dx,x\rangle} \mathrm{dvol}(x) = \int_{B_m} e^{-\lambda_1 x_1^2 - \cdots - \lambda_n x_n^2} \mathrm{dvol}(x_1, \ldots, x_n) = \prod_{i=1}^n \int_{-m}^m e^{-\lambda_i x^2} \mathrm{d}x. \tag{2}$$

By the substitution $u = \sqrt{\lambda_i} x$, we find for the latter integrals, considering Example 13.69,

$$\lim_{m\to\infty} \int_{-m}^m e^{-\lambda_i x^2} \mathrm{d}x = \frac{1}{\sqrt{\lambda_i}} \lim_{m\to\infty} \int_{-\sqrt{\lambda_i} m}^{\sqrt{\lambda_i} m} e^{-u^2} \mathrm{d}u = \frac{\sqrt{\pi}}{\sqrt{\lambda_i}}.$$

Since $\prod_{i=1}^n \lambda_i = \det(D) = \det(A)$, it follows from (2) that

$$\lim_{m\to\infty} \int_{B_m} e^{-\langle Dx,x\rangle} \mathrm{dvol}(x) = \frac{\pi^{n/2}}{\sqrt{\det(A)}}.$$

Theorem 13.74 now states that $x \mapsto e^{-\langle Dx,x\rangle}$ is improperly integrable over $\mathbb{R}^n$, and together with (1), we conclude the desired result

$$\int_{\mathbb{R}^n} e^{-\langle Ax,x\rangle}\mathrm{dvol}(x) = \int_{\mathbb{R}^n} e^{-\langle Dx,x\rangle}\mathrm{dvol}(x) = \lim_{m\to\infty}\int_{B_m} e^{-\langle Dx,x\rangle}\mathrm{dvol}(x)$$
$$= \frac{\pi^{n/2}}{\sqrt{\det(A)}}.$$

Solution of 9.7: We directly prove the general case, notice that:

$$\Phi(f(x)) = \int_0^{f(x)} \Phi'(t)\,dt = \int_{\mathbb{R}} \mathbf{1}_{[0,f(x)]}(t)\Phi'(t)\,dt, \qquad \text{for all } x \in \mathbb{R}^n.$$

We integrate this identity over $\mathbb{R}^n$ and use Fubini, to this end we reshuffle the domain noticing that

$$\mathbf{1}_{[0,f(x)]}(t) = \mathbf{1}_{[0,\infty)}(t)\cdot\mathbf{1}_{[0,\infty)}(f(x)-t) = \mathbf{1}_{[0,\infty)}(t)\cdot\mathbf{1}_{[t,\infty)}(f(x))$$

obtaining

$$\begin{aligned}
\int_{\mathbb{R}^n}\Phi(f(x))\,dx &= \int_{\mathbb{R}^n}\Big\{\int_{\mathbb{R}}\mathbf{1}_{[0,f(x)]}(t)\Phi'(t)\,dt\Big\}\,dx\\
&= \int_{\mathbb{R}^n}\Big\{\int_{\mathbb{R}}\mathbf{1}_{[0,\infty)}(t)\cdot\mathbf{1}_{[t,\infty)}(f(x))\Phi'(t)\,dt\Big\}\,dx\\
&= \int_{\mathbb{R}}\Big\{\int_{\mathbb{R}^n}\mathbf{1}_{[0,\infty)}(t)\cdot\mathbf{1}_{[t,\infty)}(f(x))\Phi'(t)\,dx\Big\}\,dt\\
&= \int_0^\infty \Phi'(t)\Big\{\int_{\mathbb{R}^n}\mathbf{1}_{[t,\infty)}(f(x))dx\Big\}\,dt\\
&= \int_0^\infty \Phi'(t)\,\mu_n(\{f(x)\geq t\})\,dx\,dt.
\end{aligned}$$

The first formula follows setting $\Phi(t) := t^p$.