Exercise sheet 0

1. Let K be a field, $\mathcal{I}$ be a set of indices and for every $i \in \mathcal{I}$ let $K_i \subset K$ be a subfield. Show that $\bigcap_{i\in\mathcal{I}} K_i$ is a subfield of K.

2. Decide which of the following quotient rings are isomorphic to each other:

(a) $R_1 := \mathbb{R}[X,Y]/(X^2)$
(b) $R_2 := \mathbb{R}[X,Y,Z]/(X,Y)$
(c) $R_3 := \mathbb{R}[X,Y,Z]/(Y^2, X+Z)$
(d) $R_4 := \mathbb{R}[X,Y]/(X+Y)$
(e) $R_5 := \mathbb{R}[X,Y,Z]/(XY)$
(f) $R_6 := \mathbb{R}[X,Y,Z]/(XY+2X+Y+2)$

3. Let R be a commutative ring and let $\varphi : R \hookrightarrow \mathbb{Z}$ be a surjective ring homomorphism. Prove that the following statements are true or give a counter-example:

(a) $\mathrm{image}(\varphi)$ is a prime ideal in $\mathbb{Z}$.
(b) For $\mathfrak{s}$ a prime ideal in $\mathbb{Z}$, $\varphi^{-1}(\mathfrak{s})$ is also a prime ideal in R.
(c) For $\mathfrak{r}$ a prime ideal in R with $\ker(\varphi) \subseteq \mathfrak{r}$, $\varphi(\mathfrak{r})$ is also a prime ideal in $\mathbb{Z}$.

4. Show that any finite integral domain is a field.

5. Let R and S be rings with 1 and $\varphi : R \to S$ be a nonzero map which satisfies $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$, $\forall a, b \in R$. Show that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is a zero divisor. Hence if S has no zero divisors then $\varphi(1_R) = 1_S$.

6. Let $\varphi : R \to Q$ be a surjective ring homomorphism. Prove that there is a one-to-one correspondence between the ideals of Q and the ideals of R that contain $\ker(\varphi)$.