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Exercise sheet 1

1. Let R be a principal ideal domain.

(a) Show that every ascending chain of ideals, $I_1 \subseteq I_2 \subseteq \cdots$, eventually become stationary. Or in other words, there is a positive index n such that $I_k = I_n$ for all $k \geqslant n.$

(b) Show that every irreducible element is a prime element.

2. Show that every principal ideal domain is a unique factorization domain.

3. Consider the ring $R := \mathbb{Z}[i] \subset \mathbb{C}$ with the so called *field norm*

$$N\colon R \to \mathbb{Z}_{\geqslant 0},\ a + bi \mapsto (a+bi)(a-bi) = a^2 + b^2.$$

(a) Prove that R is a Euclidean ring with respect to N.

(b) Determine $\gcd(3-i, 3+i)$ and $\gcd(2-i, 2+i)$ in R.

(c) Write $3+i$ as a product of prime elements from R.

(d) Prove that each prime element of R divides exactly one prime number $p \in \mathbb{Z}$.

(e) Prove that each prime number $p \equiv 3 \pmod 4$ is a prime element of R.

4. (a) Let R be a ring with unique factorization. Prove: if $a, b, c \in R$ are nonzero, $ab = c^n$ and a and b are relatively prime then there are units $u, v \in R$ as well as elements $a', b' \in R$, such that $a = ua'^n$ and $b = vb'^n$.

(b) There are counterexamples to the conclusion of (a) if we drop the the hypothesis that R has unique factorization. Use $R = \mathbb{Z}[\sqrt{-26}]$ to give such a counterexample.