Exercise sheet 10

1. Let $L : K$ be a finite Galois extension. Take $x \in L$ and assume that the elements $\sigma(x)$ are all distinct for $\sigma \in \mathrm{Gal}(L : K)$. Show: $L = K(x)$.

2. For p an odd prime number, let $\zeta := e^{2\pi i/p}$. Denote by C_i a cyclic group of order i.

(a) Show: $[\mathbb{Q}(\zeta) : \mathbb{Q}] = p - 1$. (*Hint:* Use Eisenstein criterion.)

(b) Show: $\mathrm{Gal}(\mathbb{Q}(\zeta) : \mathbb{Q}) \cong C_{p-1}$.

3. Let L_f be the splitting field of $f = X^5 - 1$ over $\mathbb{Q}$.

(a) Determine $\mathrm{Gal}\,(L_f : \mathbb{Q})$.

(b) Determine all intermediate bodies M with $\mathbb{Q} \subsetneq M \subsetneq L_f$.

(c) Let $\zeta := e^{\frac{2\pi i}{5}}$. Determine the minimum polynomial of $\zeta + \zeta^4$ over $\mathbb{Q}$.

4. For $n \geqslant 3$ let $\zeta \in \mathbb{C}$ be the primitive n-th root of unity. Prove:

$$\mathbb{Q}(\zeta) \cap \mathbb{R} = \mathbb{Q}(\zeta + \zeta^{-1})$$

and determine the degree $[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta + \zeta^{-1})]$.

5. Let K be a field, where the characteristic of K is not 2 and let $f(x) \in K[x]$, such that the zeros of f in a splitting field are $\alpha_1, \ldots, \alpha_n$. Let

$$\delta = \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j).$$

The *discriminant* $\Delta(f)$ of f is defined as

$$\Delta(f) = \delta^2.$$

Prove:

(a) $\Delta(f) \in K$.

(b) $\Delta(f) = 0$ if and only if it has a multiple zero.

(c) If $\Delta(f) \neq 0$, then $\Delta(f)$ is a perfect square in K if and only if the Galois group of f, interpreted as a group of permutations of the zeros of f, is contained in the alternating group A_n.