Exercise sheet 12

1. Let V be vector space of dimension n over the field F, let $A, B \in \mathrm{Mat}_{n\times n}(F)$ be matrices corresponding to two linear transformations on V. Let V_A and V_B be the vector space V viewed as an $F[X]$ module using A and B respectively. i.e. the action of $x \in F[X]$ on $v \in V$ is defined as $X \cdot v := Av$ (or $X \cdot v = Bv$).

Show that V_A is isomoprhic to V_B as $F[X]$ modules if and only if $B = UAU^{-1}$ for some matrix $U \in \mathrm{GL}(n, F)$.

2. Let R be a non-zero commutative ring with $0 \neq 1$. Show that if $R^n \simeq R^m$ as R-modules then $m = n$.

3. Let R be a ring, let M be an R-module and let N be a submodule of M. Prove:

(a) If M is finitely generated, then M/N is finitely generated.

(b) If N and M/N are finitely generated, then M is finitely generated.

(c) If N and M/N are free R-modules, then M is a free R-module.

4. Let R be a PID. Show that every submodule N of a free R-module M of rank n is finitely generated with at most n generators.

Hint: Apply Exercise **3**.

5. Let R be a commutative ring. An R-module M is called a *Noetherian* R-module if it satisfies the ascending chain condition on submodules, i.e., whenever

$$M_1 \subset M_2 \subset \cdots$$

is an increasing chain of submodules of M, then there is a positive integer m such that for all $k \geqslant m$ we have $M_k = M_m$.

Show that the following are equivalent for an R module M:

(a) M is a Noetherian R-module.

(b) Every non empty subset of modules of M contains a maximal element under inclusion.

(c) Every submodule of M is finitely generated.

6. Show that if R is a PID then every nonempty set of ideals of R has a maximal element and that R is Noetherian.