Exercise sheet 2

1. Decide whether the following polynomials are irreducible in $\mathbb{Q}[X]$.

(a) $X^3 - 3X^2 - 8$

(b) $2X^{10} - 25X^3 + 10X + 30$

(c) $2X^3 + X^2 + 2X + 2$

(d) $X^4 + X^2 + 1$

2. Consider the ring $R = \mathbb{Z}[X]/(X^2 + 5)$.

(a) Show that R is an integral domain.

(b) Show that R is not a unique factorization domain.

3. Consider the ring $R := \mathbb{Z}[\sqrt{-2}]$.

(a) Show that R is a Euclidean domain with the norm function

$$N\colon R \to \mathbb{Z}_{\geqslant 0},\ a + b\sqrt{-2} \mapsto a^2 + 2b^2.$$

(b) Show that the norm N is multiplicative and hence if $r \mid s$ in $\mathbb{Z}[\sqrt{-2}]$, then $N(r)$ divides $N(s)$.

(c) Show that the only units in $\mathbb{Z}[\sqrt{-2}]$ are ± 1.

4. The goal of this exercise is to show that the only integral solutions of the diophantine equation $y^2 = x^3 - 2$ are $(x, y) = (3, 5)$ and $(3, -5)$.

(a) Show that if $x, y \in \mathbb{Z}$ satisfy $y^2 = x^3 - 2$ then x is odd.

(b) Show that if $x, y \in \mathbb{Z}$ satisfy $y^2 = x^3 - 2$ then $y + \sqrt{-2}$ and $y - \sqrt{-2}$ are relatively prime over $\mathbb{Z}[\sqrt{-2}]$

(c) Write $x^3 = y^2 + 2 = (y + \sqrt{-2})(y - \sqrt{-2})$ and use Exercise sheet 1, Question 4 (a) to write $(y + \sqrt{-2}) = (a + b\sqrt{-2})^3$ and conclude that only solutions are $(x, y) = (3, 5)$ and $(3, -5)$.