Exercise sheet 3

1. (a) Let f and g be polynomials over a field F. Show that f and g are relatively prime if and only if f and g have no common root in any extention of F.

   (b) If $f, g \in F[x]$ are distinct monic irreducible polynomials then show that they have no common roots in any extention of F.

2. Let $\overline{\mathbb{Q}} := \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$, the set of all algebraic numbers over $\mathbb{Q}$.

   (a) Show that $\overline{\mathbb{Q}}$ is a field.

   (b) Show that $\overline{\mathbb{Q}} : \mathbb{Q}$ is an infinite extention

3. Let $\mathbb{A} = \mathbb{R} \bigcap \overline{\mathbb{Q}}$. Show that $\mathbb{A}$ is countable, and conclude that there are real numbers which are transcendental.

4. Let $L : K$ be an algebraic field extension. Let K_1, K_2 be two fields with $K \subseteq K_1, K_2 \subseteq L$, such that the field extensions $K_1 : K$ and $K_2 : K$ are finite. The composite of K_1 and K_2 is defined as $K_1K_2 := K(K_1 \cup K_2)$. Show:

   (a) $[\,K_1K_2 : K_2\,] \leqslant [\,K_1 : K\,]$

   (b) $[\,K_1K_2 : K\,] \leqslant [\,K_1 : K\,] \cdot [\,K_2 : K\,]$

   (c) If $\gcd([\,K_1 : K\,], [\,K_2 : K\,]) = 1$, then equality holds in (b).

   Remark: If equality holds in (b), K_1 and K_2 are said to be *linearly disjoint* over K.