D-MATH Algebra II FS 2024
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Exercise sheet 4

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1. Let $F \subset K \subset L$ be fields. Show that $L : F$ is an algebraic extention if and only if $L : K$ and $K : F$ are algebraic.

2. Let $L : K$ be an algebraic field extention. Prove that every subring R of L which contains K is a field. Give a counter example in the case that the extention is not algebraic.

3. (a) Let F be a field and $a \in \overline{\mathbb{Q}}$ that generates a field extention of F of degree 7. Prove that a^2 generated the same extention.

(b) Prove that part **3**.a holds for 7 replaced by any odd integer.

4. Let p and q are two distinct primes. Prove that $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$ are isomorphic as vector spaces over $\mathbb{Q}$ but not as fields.

5. Let $x = \sqrt{2} + \sqrt[3]{3}$.

(a) Prove that $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$. [*Hint:* Find the minimal polynomial of $x - \sqrt{2}$ and expand]

(b) Compute the minimal polynomial of x over $\mathbb{Q}(\sqrt{2})$. [*Hint:* $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{2})] = ?$]

(c) Compute the minimal polynomial of x over $\mathbb{Q}$.