Exercise sheet 5

Exercises marked with * go beyond the standard subject matter.

1. *(Constructions with compass and ruler)* In the Euclidean plane consider the following two operations:

(a) *(ruler)* Given two points P, Q, we can draw a straight line through them.

(b) *(compass)* Given two points P, Q draw a circle whose center is one of P,Q and whose radius is equal to the distance between the given points.

Assume we are given two distinct points P_0 and P_1 in the plane. Through translation, rotation and stretching, we can assume that $P_0 = (0, 0)$ and $P_1 = (1, 0)$.

We say a point P is **constructible** if there exists a finite sequence of points

$$P_0, P_1, \ldots, P_n = P$$

in the plane with the following property. Let

$$S_j = \{P_0, P_1, \ldots, P_j\}, \text{ for } 1 \leqslant j \leqslant n.$$

For each $2 \leqslant j \leqslant n$, P_j is either

(i) the intersection of two distinct straight lines, each joining two points of S_{j-1}, or

(ii) a point of intersection of a straight line joining two points of S_{j-1} and a circle with centre a point of S_{j-1} and radius the distance between two points of S_{j-1}, or

(iii) a point of intersection of two distinct circles, each with centre a point of S_{j-1} and radius the distance between two points of S_{j-1}.

In case (iii), the centres must be different if the circles are to intersect: the radii may or may not be different.

If $P = (x, y)$ is a constructible point, we consider the extension $\mathbb{Q}(x, y) : \mathbb{Q}$ generated by x and y. In this exercise we'll prove that if $P = (x, y)$ is a constructible point, the extension $\mathbb{Q}(x, y) : \mathbb{Q}$ is finite, and $[\mathbb{Q}(x, y) : \mathbb{Q}] = 2^r$, for some non-negative integer r.

Let $P_0, P_1, \ldots, P_n = P$ be a sequence of points which satisfies the requirements of the definition of P being constructible. Let $P_j = (x_j, y_j)$, and for $1 \leqslant j \leqslant n$ let

$$F_j = \mathbb{Q}(x_1, y_1, x_2, y_2, \ldots, x_j, y_j),$$

so that $F_{j+1} = F_j(x_{j+1}, y_{j+1})$, for $1 \leqslant j < n$.

(a) Prove that $[F_{j+1} : F_j] = 1$, if (x_{j+1}, y_{j+1}) is the intersection of two distinct straight lines, each joining two points of S_j, i.e. arises from an intersection point of type (i).

(b) Prove that $[F_{j+1} : F_j] = 1$ or 2, if (x_{j+1}, y_{j+1}) is a point of intersection of an appropriate straight line and circle, i.e. arises from an intersection point of type (ii).

(c) Prove that $[F_{j+1} : F_j] = 1$ or 2, if (x_{j+1}, y_{j+1}) is a point of intersection of two circles.

(d) Show that

$$[F_n : F_1] = [F_n : \mathbb{Q}] = 2^s,$$

for some $s \in \mathbb{Z}_{\geqslant}$ and conclude

$$[\mathbb{Q}(x, y) : \mathbb{Q}] = 2^r,$$

for some non-negative integer r.

2. *(Trisecting the angle)* It is not possible to construct the angle $\alpha/3$ for every angle α by using only a finite number of steps with a compass and ruler.

Hint: Consider $\cos(\frac{\pi}{9})$ and find its minimal polynomial.

3. *(Doubling the cube)* It is impossible to construct the number $\sqrt[3]{2}$ by using only a finite number of steps with a compass and ruler.

4. *(Squaring the circle)* It is impossible to construct a square with the area of a given circle by using only a finite number of steps with a compass and ruler.

5. * *(Classical geometric constructions)* For any set A of points, let $Cons(A)$ be the set of all points that can be constructed from A (by iterated application of the operations in Exercise **1**.).

We identify the Euclidean plane with $\mathbb{C}$ with the usual distance $d(P, Q) := |P - Q|$. To be able to construct new points, we assume that A contains at least two different points. Through translation, rotation and stretching, we can assume that A contains at least the points 0 and 1.

Let $K = Cons(A)$. Prove that

(i) K is a subfield of $\mathbb{C}$

(ii) $\forall z \in K : \overline{z} \in K$

(iii) $\forall z \in \mathbb{C} : z^2 \in K \to z \in K$