Exercise sheet 6

1. Let K be a field of characteristic 0 and $L : K$ a finite algebraic extention. Show that $L : K$ is simple if and only if there are only finitely many intermediate fields.

2. (a) Prove that if $[K : k] = 2$, then $k \subseteq K$ is a normal extention.

(b) Show that $\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}$ is normal.

(c) Show that $\mathbb{Q}(\sqrt[4]{2}(1+i)) : \mathbb{Q}$ is not normal over $\mathbb{Q}$.

(d) Deduce that given a tower $L : K : k$ of field extentions, $L : k$ needs not to be normal even if $L : K$ and $K : k$ are normal.

3. (a) Let K be field containing $\mathbb{Q}$. Show that any automorphism of K is a $\mathbb{Q}$-automorphism.

(b) From now on, let $\sigma : \mathbb{R} \to \mathbb{R}$ be a field automorphism. Show that σ is increasing:

$$x \leqslant y \Longrightarrow \sigma(x) \leqslant \sigma(y).$$

(c) Deduce that σ is continuous.

(d) Deduce that $\sigma = \mathrm{Id}_{\mathbb{R}}$.

4. (a) Show that every finite field is isomorphic to $\mathbb{F}_p[x]/(f(x))$ for some prime p and some monic irreducinle polynomial $f(x)$ in $\mathbb{F}_p[x]$.

(b) Show that each irreducible polynomial $f(x)$ in $\mathbb{F}_p[x]$ of degree n divides $x^{p^n} - x$ and is separable.

(c) Factor $x^8 - x$ and $x^{16} - x$ in $\mathbb{F}_2[x]$

5. (a) Show that $(x^d - 1)|(x^n - 1)$ if and only if $d|n$

(b) Prove that a subfield F of $\mathbb{F}_{p^n}$ has order p^d where $d|n$.

(c) Show that for each $d|n$ there is one subfield F of $\mathbb{F}_{p^n}$ of order p^d.