D-MATH | Algebra II | FS 2024

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Exercise sheet 8

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1. Recall that a normal closure of an extention $L : K$ is the smallest extention of L which is normal over K. Let $L : K$ be a finite extention. Show that there exists a normal closure N of $L : K$ which is a finite extention of K and that if M is another normal closure than the extentions $M : K$ and $N : K$ are isomorphic.

Hint: Let $\alpha_1, \dots \alpha_n$ be a basis of L over K with minimal polynomials $m_i = m_{\alpha_i,K}$ and consider the splitting field of the polynomial $m_1 m_2 \dots m_n$.

2. Let $L : K$ be a finite extention. Show that the following are equivalent

(a) $L : K$ is normal

(b) For every finite extention M of K containing L, every K-monomorphism $\varphi : L \to M$ is a K-automorphism of L.

(c) There exists a finite normal extention N of K containing L such that every every K-monomorphism $\varphi : L \to N$ is a K-automorphism of L.

3. Let $L : K$ be a separable, finite extention of degree n. Show that there are exactly n K-monomorphisms of L into a normal closure N.

4. Show that $x^4 + 1$ is irreducible in $\mathbb{Z}[x]$ but reducible in $\mathbb{F}_p[x]$ for every prime p.

5. Let L be the splitting field of the polynomial $x^4 + 1$ over $\mathbb{Q}$ and let $G = \text{Gal}(L : \mathbb{Q})$ be its Galois group. Determine G and the fixed fields corresponding to each of its subgroups.