Exercise sheet 9

1. (a) Let p be a prime and $f \in \mathbb{Q}[x]$ an irreducible polynomial of degree p with splitting field L. Assume that f has exactly $p-2$ real roots. Show that $\mathrm{Gal}(L:\mathbb{Q}) \simeq S_p$.

Hint: Make use of the following two facts from the theory of finite groups.

i. (Cauchy) If G is a finite group and p is a prime with $p \mid |G|$ then G contains an element of order p.

ii. A $p-$cycle $(a_1, a_2 \cdots a_n)$ with $\{a_1, \ldots a_n\} = \{1, 2, \ldots, n\}$ and a transposition (a_i, a_j) where generate the group S_p.

(b) Show that the Galois group of $x^5 - 4x + 2 \in \mathbb{Q}[x]$ is isomorphic to S_5

2. Let $L : K$ be a finite separable extention. Use Galois theory to show that there are finitely many intermediate fields between L and K. Use Question 1 of Serie 6 to conclude that $L : K$ is simple.

3. Determine the Galois group of $x^6 - 8$ over $\mathbb{Q}$.

4. Let $f(X) \in \mathbb{Q}[X]$ be a non zero polynomial. Assume that the order of the Galois group of $f(x)$ over $\mathbb{Q}$ is odd. Prove that all zeros of $f(x)$ are real.

5. Let $L : K$ be a finite Galois extension with intermediate fields K_1, K_2 and corresponding Galois groups $G_i := \mathrm{Gal}(L : K_i) \leqslant G := \mathrm{Gal}(L : K)$. Prove:

(a) $K_1 K_2 = L^{G_1 \cap G_2}$

(b) $K_1 \cap K_2 = L^{\langle G_1, G_2 \rangle}$, where $\langle G_1, G_2 \rangle$ is the subgroup of G generated by G_1 and G_2

(c) If $K_1 K_2 = L$, $K_1 \cap K_2 = K$ and the extensions $K_1 : K$ and $K_2 : K$ are both Galois, then

$$\mathrm{Gal}(L : K) \cong G_1 \times G_2$$

Hint: If G is a group with two normal subgroups G_1 and G_2 such that $G_1 \cap G_2 = 1$, then $G_1 G_2 \cong G_1 \times G_2$.