Single Choice 13

1. Let R be a ring and M an R-module. For $n \in \mathbb{Z}_{\geqslant 1}$ and each $1 \leqslant i \leqslant n$ let M_i be a submodule of M. Which of the following statements is **false**?

(a) The sum $\sum_{i=1}^{n} M_i$ is a submodule of M.
(b) The direct sum $\bigoplus_{i=1}^{n} M_i$ is a submodule of M^n.
(c) The intersection $\bigcap_{i=1}^{n} M_i$ is a submodule of M.
(d) The union $\bigcup_{i=1}^{n} M_i$ is a submodule of M.

2. Let R be a ring. Which of the following statements is **false**?

(a) Each submodule of R is an ideal.
(b) Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}$ is a submodule of R.
(c) For each ideal $\mathfrak{a} \subset R$, $R/\mathfrak{a}$ is an R-module.
(d) Let M and N be two R-modules generated by a single element. Then $M \cong N$.

3. Let M and N be two $\mathbb{Z}$-modules. Which of the following statements is **false**?

(a) A $\mathbb{Z}$-module homomorphism is an isomorphism if it is bijective.
(b) For each $M \twoheadrightarrow N$ surjective $\mathbb{Z}$-module homomorphism there exists a submodule $\tilde{M}$ of M such that $\tilde{M} \cong N$.
(c) For each $M \twoheadrightarrow N$ surjective $\mathbb{Z}$-module homomorphism there exists a submodule $\tilde{M}$ of M such that $M/\tilde{M} \cong N$.
(d) There exists a $\mathbb{Z}$-module homomorphism $M \to N$.

4. Let $R := \mathbb{Z}[\sqrt{-5}]$. Let $\mathfrak{p} := (3, 1 + \sqrt{-5})$ and $\mathfrak{q} := (3, 1 - \sqrt{-5})$ be ideals of R. Which of the following statements is true?

(a) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as $\mathbb{Z}$-modules, but not as R-modules.
(b) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as R-modules, but not as $\mathbb{Z}$-modules.
(c) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as both R-modules and $\mathbb{Z}$-modules.
(d) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are not isomorphic as either R-modules or $\mathbb{Z}$-modules.

5. Consider the Q-module $M := \mathbb{Q}^2$ as a $\mathbb{Q}[X]$-module such that scalar multiplication by X is given by left multiplication by the matrix $A := \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$. Which of the following $\mathbb{Q}[X]$-isomorphisms holds?

(a) $M \cong \mathbb{Q}[X]/(X - 9)$
(b) $M \cong \mathbb{Q}[X]/(X^2 - 9)$
(c) $M \cong \mathbb{Q}[X]/(X)$
(d) $M \cong \mathbb{Q}[X]/(X + 3)^2$