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Review Multiple Choice

1. Consider the ring $R = \mathbb{Z}[i\sqrt{2}, \frac{1}{2}]$. Which of the following is a subring of R?

(a) $2\mathbb{Z}$

(b) $\mathbb{Z}[i]$

(c) $\{a \cdot i\sqrt{2} \mid a \in \mathbb{Z}\}$

(d) $\mathbb{Z}[\frac{i}{\sqrt{2}}]$

2. Let $L : K$ be a finite extension of fields. Which of the following assertions are correct:

(a) If the characteristic of K is zero, then $L : K$ is normal.

(b) If the characteristic of K is zero, then $L : K$ is separable.

(c) If $L : K$ is normal, then $L : K$ is a Galois extension.

(d) If the characteristic of K is positive, then $L : K$ is normal if and only if it is separable.

3. Let K be a field, $\bar{K}$ an algebraic closure of K and $P \in K[X]$ a non-constant polynomial. Let $L \subset \bar{K}$ denote the splitting field of P in $\bar{K}$. Which of the following assertions are correct:

(a) The extension $L : K$ is a normal extension.

(b) If $x \in \bar{K}$ is a root of P, then $L = K(x)$.

(c) The extension $L : K$ is a Galois extension.

(d) If the polynomial P is irreducible, then $L : K$ is a Galois extension.

(e) If the characteristic of K is zero, then $L : K$ is a Galois extension.

4. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K such that $L : K$ is a Galois extension. Let $K \subset E \subset L$ be an intermediate extension. Which of the following assertions are correct:

(a) The extension $L : E$ is a Galois extension.

(b) The extension $E : K$ is a normal extension.

(c) The extension $E : K$ is a separable extension.

5. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K such that $L : K$ is a Galois extension, and let G be its Galois group. Which of the following assertions are correct:

(a) For any subgroup H of G, the intermediate extension $E = L^H$ is a normal extension of K.

(b) Two subgroups H_1 and H_2 of G are equal if and only if $L^{H_1} = L^{H_2}$.

(c) Any subgroup H of G is the Galois group of some extension $E : K$ for some $E \subset L$.

(d) Any subgroup H of G is the Galois group of some extension $L : E$ for some $E \subset L$.

6. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K such that $L : K$ is a Galois extension, and let G be its Galois group. Let $x \in L$ be given and $\sigma_0 \in G$ a non-trivial element. Which of the following assertions are correct:

(a) If $\sigma_0(x) = x$, then $x \in K$.

(b) If G is cyclic and $\sigma_0(x) = x$, then $x \in K$.

(c) The element

$$\sum_{\sigma \in G} \sigma(x)^2$$

belongs to K.

7. Let R be a ring and $\mathfrak{a} \subsetneq R$ an ideal. Which of the following statements are true?

(a) For arbitrary $r, s \in R$ we have $r + \mathfrak{a} = s + \mathfrak{a}$ if and only if $r = s$.

(b) If there exists a field K and a ring homomorphism $\varphi : R \to K$, such that $\ker(\varphi) = \mathfrak{a}$, then $\mathfrak{a}$ is a maximal ideal.

(c) We have $(x) + \mathfrak{a} = (1)$ for all $x \in R \backslash \mathfrak{a}$ if and only if $\mathfrak{a}$ is maximal.

(d) If $\mathfrak{a} = (a, b)$ for some elements $a, b \in R$, then $\mathfrak{a}$ is not principal.

8. Consider the ring $R := \mathbb{Z}[i]$ which, with respect to the norm mapping

$$N : R \to \mathbb{Z}^{>0} : a + bi \mapsto a^2 + b^2$$

is a Euclidean ring. Which of the following statements are correct?

(a) The element 2 is prime in R.

(b) An element $\pi \in R$ is irreducible if and only if $N(\pi)$ is prime.

(c) For any $r_1, \ldots, r_n \in R$ there are elements $x_1, \ldots, x_n \in R$ such that

$$\gcd(r_1, \ldots, r_n) = x_1 r_1 + \ldots + x_n r_n.$$

(d) $\gcd(4 + i, 3 + 5i) \sim 1 - 4i$.

9. Which of the following polynomials is irreducible?

(a) $\frac{1}{10}X^4 + 3X^3 + 15X + \frac{2}{10}$ in the ring $\mathbb{Q}[X]$.

(b) $X^{2016} + X^{19} + X^2 - 1$ in the ring $\mathbb{Z}/3\mathbb{Z}[X]$.

(c) $Y^3 + (X^2 - 2iX - 1)\,Y^2 + (X^2 + 1)\,Y - X + i$ in the ring $\mathbb{C}[X, Y]$

10. Consider a ring R, a field K, and a homomorphism $\varphi : R \to K$. Which of the following statements are true?

(a) Then $\ker(\varphi)$ is a prime ideal of R.

(b) If K is finite, R must also be finite.

(c) If R is finite, $\mathrm{im}(\varphi)$ must be a field.