Solutions Single Choice 1

1. Which of the following statements is true?

(a) In any ring R, the zero ideal (0) is a prime ideal.

(b) Each principal ideal domain is an Euclidean domain.

(c) For all fields K and M, each ring homomorphism $K \to M$ is injective.

(d) For all fields K and M, each ring homomorphism $K \to M$ is surjective.

Solution: The correct answer is (c). The zero ideal is prime if and only if the ring is an integral domain. In (b), the converse statement is true, $\mathbb{Z}[(1+\sqrt{-19})/2]$ is a PID which is not Euclidean. For (d) we can take $\mathbb{Q} \hookrightarrow \mathbb{R}$.

2. Which subring of $\mathbb{C}$ is **not** equal to the others?

(a) $\mathbb{Z}[\frac{i}{20}, -10i]$

(b) $\mathbb{Z}[\frac{1}{2}, \frac{i}{5}]$

(c) $\mathbb{Z}[\frac{1}{2}, 10i]$

(d) $\mathbb{Z}[\frac{i}{20}, \frac{10}{i}]$

Solution: From $\frac{10}{i} = -10i$ we see immediately that the rings given in (a) and (d) are equal. Further we have $\frac{1}{2} = \frac{i}{20} \cdot (-10i)$ and $\frac{i}{5} = 4 \cdot \frac{i}{20}$, so that the ring $\mathbb{Z}[\frac{1}{2}, \frac{i}{5}]$ is contained in $\mathbb{Z}[\frac{i}{20}, -10i]$. On the other hand, we have $\frac{i}{20} = \left(\frac{1}{2}\right)^2 \cdot \frac{i}{5}$ and $-10i = -50 \cdot \frac{i}{5}$, so that the rings given in (a) and (b) are equal. Since $10i = 50 \cdot \frac{i}{5}$, we have $\mathbb{Z}[\frac{1}{2}, 10i] \subset \mathbb{Z}[\frac{1}{2}, \frac{i}{5}]$. We claim that $\frac{1}{25} \notin \mathbb{Z}[\frac{1}{2}, 10i]$. Each element of $\mathbb{Z}[\frac{1}{2}, 10i]$ can be represented as a finite sum

$$\sum_i a_i \left(\frac{1}{2}\right)^{b_i} (10i)^{c_i},$$

for $a_i \in \mathbb{Z}, b_i, c_i \in \mathbb{Z}_{\geqslant 0}$. Thus we can never get a denominator divisible by 5, so our claim is true. But $\frac{1}{25} = -\left(\frac{i}{5}\right)^2 \in \mathbb{Z}[\frac{1}{2}, \frac{i}{5}]$, hence the correct answer is (c).

3. $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is a group under componentwise addition. Consider the subgroup

$$H := \{h \cdot (1,2,3) \mid h \in \mathbb{Q}\}.$$

Then $(\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q})/H$ is isomorphic to

(a) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$

(b) $\mathbb{Q} \times \mathbb{Q}$

(c) $\mathbb{Q}$

(d) $\{0\}$

Solution: Define $f : \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ by $f(x, y, z) = (y - 2x, z - 3x)$.

Note that for each vector $\underline{x} \in H$ we have that $f(\underline{x}) = \underline{0}$, so that $H \subseteq \ker(f)$. On the other hand, let $(x, y, z) \in \ker(f)$. Then $f(x, y, z) = (0, 0) \iff (y - 2x, z - 3x) = (0, 0)$. Thus, we have $y = 2x$ and $z = 3x$. Thus $(x, y, z) = (x, 2x, 3x) \in H$.

Let $(a, b) \in \mathbb{Q} \times \mathbb{Q}$. Then $f(0, a, b) = (a, b)$, so f is surjective. Hence we conclude that (b) is the correct answer using the first isomorphism theorem.

4. Which of the following ideals in $\mathbb{Q}[X]$ is not a maximal ideal?

(a) $(X + 1)$

(b) $(X^2 + 1)$

(c) $(X^3 + 1)$

(d) $(X^4 + 1)$

Solution: An ideal (f), for $0 \neq f \in \mathbb{Q}[X]$ is maximal if and only if f is irreducible. Since the only reducible polynomial above is $X^3 + 1 = (X + 1)(X^2 - X + 1)$, the correct answer is (c).

5. Let R^* be the group of units. Which of the following is true for each $x \in R^*$?

(a) $x + 1 \in R^*$

(b) $x^2 \in R^*$

(c) $\forall y \in R \, \exists z \in R : y \cdot z = x$

(d) All the statements above are true.

Solution: Only (b) is correct. A product of two elements in R^* is again contained in R^*. In a) we can take $x = -1$, and in c) take $y = 0$ as counter examples.