Solutions Single Choice 10

1. The group $\mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q})$ is isomorphic to...

(a) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

(c) $\mathbb{Z}/8\mathbb{Z}$

(d) D_4

Solution: The correct answer is (d). Since $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ and $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})] = 2$ (since $\mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$), we have that the Galois group $G := \mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q})$ has order 8. Let $\sigma \in G$ be the automorphism sending $\sqrt[4]{2} \mapsto i\sqrt[4]{2}$ and $i \mapsto i$. Let $\theta \in G$ denote complex conjugation. Then σ has order 4. Since $\theta\sigma\theta^{-1}$ sends $\sqrt[4]{2}$ to $-i\sqrt[4]{2}$ and i to i, we have $\theta\sigma\theta^{-1} = \sigma^3 = \sigma^{-1}$, so that

$$\langle \sigma, \theta \rangle = \langle \sigma, \theta \mid \sigma^4 = \theta^2 = 1, \theta\sigma\theta^{-1} = \sigma^{-1} \rangle \cong D_4.$$

Since $|D_4| = 8$ and the order of G is 8, we are done.

2. Let $L : K$ be a Galois extension with Galois group G. Let $a \in L$ be given. Which of the following statements is false?

(a) Let $\sigma \in G$ be a non-trivial element. If G is cyclic and $\sigma(a) = a$, then $a \in K$.

(b) The element
$$\sum_{\varphi \in G} \varphi(a)$$
is in K.

(c) If the set $\{\varphi(a) \mid \varphi \in G\}$ contains at most two elements, then $[K(a) : K] \leqslant 2$.

(d) If $\varphi(a) = a$ for all $\varphi \in G$, then $a \in K$.

Solution: The correct answer is (a). We know that part (d) is true from the lectures. For part (a), note that σ might not be a generator of the cyclic group.

For part (b), let $\tau \in G$ be arbitrary. Then

$$\tau\left(\sum_{\varphi \in G} \varphi(a)\right) = \sum_{\varphi \in G} \tau(\varphi(a)) = \sum_{\tilde{\varphi} \in G} \tilde{\varphi}(a),$$

so by part (d), we obtain that this is in K.

For part (d), note that the roots of the minimal polynomial f of a are among the values $\varphi(a)$, by transitivity of the action of the Galois group of the splitting field of f on the set of roots.

3. Which of the following extensions is a normal closure of $\mathbb{Q}(\sqrt[4]{5}) : \mathbb{Q}$?

(a) $\mathbb{Q}(\sqrt[8]{5}, i\sqrt[4]{5}) : \mathbb{Q}(\sqrt[4]{5})$

(b) $\mathbb{Q}(\sqrt[4]{5}, i\sqrt[2]{5}) : \mathbb{Q}(\sqrt[4]{5})$

(c) $\mathbb{Q}(\sqrt[8]{5}, i) : \mathbb{Q}(\sqrt[4]{5})$

(d) $\mathbb{Q}(\sqrt[4]{5})$ is already a normal closure of $\mathbb{Q}(\sqrt[4]{5}) : \mathbb{Q}$

Solution: The correct answer is (b): the minimal polynomial of $\sqrt[4]{5}$ over $\mathbb{Q}$ is X^4-5, and it has zeros $\pm\sqrt[4]{5}$ and $i\sqrt[4]{5}$. Hence $\mathbb{Q}(\sqrt[4]{5}, i\sqrt[4]{5})$ is a normal closure. Since $i = i\sqrt[2]{5}/(\sqrt[4]{5})^2$, we have that $\mathbb{Q}(\sqrt[4]{5}, i\sqrt[4]{5}) = \mathbb{Q}(\sqrt[4]{5}, i\sqrt[2]{5})$. Since $\mathbb{Q}(\sqrt[4]{5}, i\sqrt[2]{5}) \subsetneq \mathbb{Q}(\sqrt[8]{5}, i\sqrt[4]{5})$ and $\mathbb{Q}(\sqrt[4]{5}, i\sqrt[2]{5}) \subsetneq \mathbb{Q}(\sqrt[8]{5}, i)$ and the normal closure is the smallest extension of $\mathbb{Q}(\sqrt[4]{5})$ which is normal, (a) and (c) are wrong.

4. Let $L : K$ be a Galois extension with Galois group G. Which of the following statements is true?

(a) Any subgroup $H \leqslant G$ is the Galois group of some extension $M : K$, for some $M \subset L$.

(b) Any subgroup $H \leqslant G$ is the Galois group of some extension $L : M$, for some $M \subset L$.

(c) For any subgroup $H \leqslant G$ the intermediate extension L^H is a normal extension of K.

(d) None of the statements above is true.

Solution: Part (a) is not correct, a counterexample is given by: the symmetric group $G = S_3$ and H is generated by a cycle of length 3, so that $|H| = 3$, then an intermediate M with $\mathrm{Gal}(M : K) = H$ would correspond to a normal subgroup $R < G$ with $[S_3 : R] = [L : M] = 2$, but one can see easily that there is no normal subgroup of order 2 in S_3. We have seen part (b) in the lectures in the Galois correspondence. For part (c), note that L^H is normal over K if and only if H is a normal subgroup of K.

5. Which of the following statements is true for every algebraic field extensions $M : L : K$?

(a) If $M : K$ is Galois, then also $M : L$ is Galois.

(b) If $M : K$ is Galois, then also $L : K$ is Galois.

(c) If $M : L$ and $L : K$ are both Galois, then also $M : K$ is Galois.

(d) All of the statements above are true.

Solution: The correct answer is (a): this is Theorem 4.3 from the lectures. A counter-example for part (b) is $\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}$, and for part (c) consider $\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2}) : \mathbb{Q}$.