Solutions Single Choice 11

1. The Galois group of the splitting field of the polynomial $x^3 + x + 1$ over $\mathbb{Q}$ is isomorphic to...

(a) C_2
(b) C_6
(c) A_3
(d) S_3

Solution: The correct answer is (d). Each rational zero of the polynomial $f(x) := x^3 + x + 1$ has to divide the constant term, but since ± 1 are not zeros, we have that f has no rational zeros. Since it has degree 3, it has to be irreducible over $\mathbb{Q}$.

Since the polynomial has order 3, its Galois group G is a subgroup of S_3. By looking at the graph of the function $\mathbb{R} \to \mathbb{R}, x \mapsto x^3 + x + 1$, we see that it has precisely one real zero, which we will denote by a. Then $[\mathbb{Q}(a) : \mathbb{Q}] = 3$, as f is irreducible, and since the other two zeros are complex, those complex zeros generate a field extension of degree 2 over $\mathbb{Q}(a)$. Hence the splitting field of f over $\mathbb{Q}$ has degree 6, and its Galois group is equal to S_3.

2. The order of the Galois group of the splitting field of the polynomial $x^3 - 2x + 1$ over $\mathbb{Q}$ is equal to...

(a) 2
(b) 3
(c) 6
(d) 9

Solution: The correct answer is (a). We can factor $x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$, so it has zeros 1 and $\frac{-1 \pm \sqrt{5}}{2}$. Hence the splitting field of $x^3 - 2x + 1$ has degree 2 over $\mathbb{Q}$ and the Galois group has order 2.

3. Let $f(x) := x^3 - 12x + 34$. Let K be a splitting field of f over $\mathbb{Q}$. Which of the following statements is **false**?

(a) f is irreducible over $\mathbb{Q}$
(b) $\mathrm{Gal}(K : \mathbb{Q})$ contains an element of order 2
(c) f has precisely one real zero
(d) $\mathrm{Gal}(K : \mathbb{Q}) \cong A_3$

Solution: The correct answer is (d). Using Eisenstein's criteria for $p = 2$, we obtain that f is irreducible over $\mathbb{Q}$.

To see that f has precisely one real root, we can compute its derivative: $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$. Since $(f(-2), f'(-2)) = (-18, 0)$ and $(f(2), f'(2)) = (-50, 0)$ we get that f increases from $-\infty$ to -2, then decreases to -50 on the interval $(-2, 2)$, and then steadily increases until it passes the real line (and thus has only one real zero). Hence part (c) follows.

Since f has two complex zeros, we have that complex conjugation has to have order 2 in the Galois group, so part (b) is true.

The Galois group of f over $\mathbb{Q}$ has to have an element of order 2, so the order of the group is also divisible by 2 and as $|A_3| = 3$, part (d) is false.

Remark. In both Exercises 1 and 3 we could have used the discriminant as an alternative way to solve them.

4. Let M be a Galois extension of K with $|\operatorname{Gal}(M : K)| = 12$. Which of the following statements is **false**?

(a) There always exists a subfield L of M containing K with $[L : K] = 2$.
(b) There always exists a subfield L of M containing K with $[L : K] = 3$.
(c) There always exists a subfield L of M containing K with $[L : K] = 4$.
(d) There always exists a subfield L of M containing K with $[L : K] = 6$.

Solution: The correct answer is (a). let d be a positive integer. By the Galois correspondence, there exists a subfield L of M containing K with $[L : K] = d$ if and only if there exists a subgroup $H \leqslant \operatorname{Gal}(M : K) =: G$ with $|G|/|H| = d$.

The group A_4 has order 12, but it does not have a subrgoup of order 6. This means that if $G \cong A_4$, then there does not exist a subgroup $H \leqslant G$ with $|G|/|H| = 12/6 = 2$, so there does not exist a subfield L of M containing K with $[L : K] = 2$.

Parts (b), (c) and (d) follow from the fact that every group of order 12 has a subgroup of order $2, 3$ and 4. This follows from the Sylow theorems.

5. Which of the following statements is **false**?

(a) Let K be a field, $f \in K[X]$ a polynomial, such that the discriminant of f is non-zero and a perfect square. Then the Galois group of f over K is contained in A_n for $n = \deg(f)$.
(b) If $K : \mathbb{Q}$ is a finite normal extension, then there exists an extension N of K such that every $\mathbb{Q}$-monomorphism $\varphi : K \to N$ is a $\mathbb{Q}$-automorphism of K.
(c) Every finite, separable and normal field extension is Galois.
(d) The 11-th cyclotomic polynomial is equal to $\sum_{k=0}^{10} x^k$.

Solution: The correct answer is (a): this holds true if $\operatorname{char}(K) \neq 2$ (see Exercise sheet 10, question 5). Part (b) is Theorem 4.8 from the lectures. Part (c) is Theorem 4.10. For part (d), we have seen in the lectures that for a prime number p, the p-th cyclotomic polynomial is given by $\sum_{k=0}^{p-1} x^k$.