Solutions Single Choice 13

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1. Let R be a ring and M an R-module. For $n \in \mathbb{Z}_{\geqslant 1}$ and each $1 \leqslant i \leqslant n$ let M_i be a submodule of M. Which of the following statements is **false**?

(a) The sum $\sum_{i=1}^{n} M_i$ is a submodule of M.

(b) The direct sum $\bigoplus_{i=1}^{n} M_i$ is a submodule of M^n.

(c) The intersection $\bigcap_{i=1}^{n} M_i$ is a submodule of M.

(d) The union $\bigcup_{i=1}^{n} M_i$ is a submodule of M.

Solution: The correct answer is (d). Part (a) was seen in the lectures. For part (b) note that addition and multiplication of the direct sum $\bigoplus_{i=1}^{n} M$ are defined componentwise. Part (c) was discussed in the lecture (see lecture notes of 22.05.). A counter-example for part (d) is given for $R \neq 0$ and $M := R^2$. Let $M_1 := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ and $M_2 := \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$. Then for example $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin M_1 \cup M_2$.

2. Let R be a ring. Which of the following statements is **false**?

(a) Each submodule of R is an ideal.

(b) Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}$ is a submodule of R.

(c) For each ideal $\mathfrak{a} \subset R$, $R/\mathfrak{a}$ is an R-module.

(d) Let M and N be two R-modules generated by a single element. Then $M \cong N$.

Solution: The correct answer is (d). Parts (a) and (b) were seen in the lectures. From Prop. 9.3 in the lectures it follows that $R/\mathfrak{a}$ is a submodule as well. For part (d), let $R := \mathbb{Z}$ and consider $M := \mathbb{Z}$ and $N := \mathbb{Z}/2\mathbb{Z}$. Both are generated by a single element, but are not isomorphic (one is finite while the other is infinite).

3. Let M and N be two $\mathbb{Z}$-modules. Which of the following statements is **false**?

(a) A $\mathbb{Z}$-module homomorphism is an isomorphism if it is bijective.

(b) For each $M \twoheadrightarrow N$ surjective $\mathbb{Z}$-module homomorphism there exists a submodule $\tilde{M}$ of M such that $\tilde{M} \cong N$.

(c) For each $M \twoheadrightarrow N$ surjective $\mathbb{Z}$-module homomorphism there exists a submodule $\tilde{M}$ of M such that $M/\tilde{M} \cong N$.

(d) There exists a $\mathbb{Z}$-module homomorphism $M \to N$.

Solution: The correct answer is (b): take $M := \mathbb{Z}$ and $N := \mathbb{Z}/2\mathbb{Z}$. Then no submodule of $\mathbb{Z}$ has an element $m \in \mathbb{Z}\backslash\{0\}$ for which we would have $2m = 0$. Part (a) was seen in the lectures. Part (c) follows by taking $\tilde{M}$ to be the kernel of the homomorphism and applying Theorem 9.4. For part (d) we can always take the zero-homomorphism.

4. Let $R := \mathbb{Z}[\sqrt{-5}]$. Let $\mathfrak{p} := (3, 1 + \sqrt{-5})$ and $\mathfrak{q} := (3, 1 - \sqrt{-5})$ be ideals of R. Which of the following statements is true?

(a) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as $\mathbb{Z}$-modules, but not as R-modules.

(b) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as R-modules, but not as $\mathbb{Z}$-modules.

(c) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as both R-modules and $\mathbb{Z}$-modules.

(d) The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are not isomorphic as either R-modules or $\mathbb{Z}$-modules.

Solution: The correct answer is (c). Complex conjugation is a $\mathbb{Z}$-module isomorphism. Since

$$\mathfrak{p} := \{3a + (1 + \sqrt{-5})b \mid a, b \in R\},$$

we have that $\overline{\mathfrak{p}} = (3, 1 - \sqrt{-5}) = \mathfrak{q}$, so $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic as $\mathbb{Z}$-modules.

Next we want to check if they are isomorphic as R-modules. Note that complex conjugation is not R-linear: let φ denote complex conjugation, and for $r \in R$ and m an element of an R-module, we would have $\varphi(rm) = \overline{r}\varphi(m)$ instead of $r\varphi(m)$. Thus we would have to construct an isomorphism differently. We will describe a more general way how to do that.

Let K be the fraction field of R. If $k \in K^\times$ then the map on R-modules $\varphi : a \mapsto \frac{1}{k}a$ is R-linear: $\varphi(rm) = \frac{1}{k}rm = \varphi(rm)$. It is also clearly additive. Moreover, φ is bijective since we can define an inverse by setting $\varphi^{-1}(b) := kb$. So if $\mathfrak{p} = k\mathfrak{q}$ for some $k \in K^\times$, we would have an isomorphism. Note that $K = \mathbb{Q}[\sqrt{-5}]$ and calculate

$$\begin{aligned} \frac{2 + \sqrt{-5}}{3}\{3a + (1 + \sqrt{-5})b \mid a, b \in R\} &= \{(2 + \sqrt{-5})a + (-1 + \sqrt{-5})b \mid a, b \in R\} \\ &= \{3a + (1 - \sqrt{-5})b \mid a, b \in R\}, \end{aligned}$$

so that $\varphi : \mathfrak{p} \to \mathfrak{q}, a \mapsto \frac{2+\sqrt{-5}}{3}a$ is an R-isomorphism.

5. Consider the Q-module $M := \mathbb{Q}^2$ as a $\mathbb{Q}[X]$-module such that scalar multiplication by X is given by left multiplication by the matrix $A := \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$. Which of the following $\mathbb{Q}[X]$-isomorphisms holds?

(a) $M \cong \mathbb{Q}[X]/(X - 9)$

(b) $M \cong \mathbb{Q}[X]/(X^2 - 9)$

(c) $M \cong \mathbb{Q}[X]/(X)$

(d) $M \cong \mathbb{Q}[X]/(X + 3)^2$

Solution: The correct answer is (b). The characteristic polynomial of the matrix A is given by $X^2 - 9 = (X - 3)(X + 3)$. Hence the polynomial $X^2 - 9$ kills everything in V, since for $\mathbf{v} \in M$, we have $(X^2 - 9)\mathbf{v} = A^2\mathbf{v} - 9\mathbf{v} = 0$. Hence

$$\mathbb{Q}[X]/(X^2 - 9) \cong M.$$