Solutions Single Choice 3

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1. Let R be an integral domain and $p \in R$. Which of the following statements is not equivalent to the others?

(a) p is prime.

(b) R/pR is an integral domain.

(c) p is irreducible.

(d) All the statements above are equivalent to each other.

Solution: The solution is (c). From the lectures we know that p is prime if and only if the quotient R/pR is an integral domain. For a counter-example for part (c) being equivalent to (a) see Question 5 below.

2. Which of the following statements is false?

(a) $(\mathbb{Z}[X])^* = \mathbb{Z}^*$

(b) $(\mathbb{Z}[i][X])^* = (\mathbb{Z}[i])^*$

(c) $(\mathbb{Z}/7\mathbb{Z}[X])^* = (\mathbb{Z}/7\mathbb{Z})^*$

(d) $(\mathbb{Z}/9\mathbb{Z}[X])^* = (\mathbb{Z}/9\mathbb{Z})^*$

Solution: The correct answer is (d). For each integral domain R we have that $(R[X])^* = R^*$. Consider the ring $\mathbb{Z}/9\mathbb{Z}[X]$. The element $1 + 3X$ is invertible, since

$$(1 + 3X)(1 - 3X) = 1 - 9X^2 = 1.$$

But, $1 + 3X \notin \mathbb{Z}/9\mathbb{Z}$.

3. Which of the following polynomials is reducible in $\mathbb{Q}[X]$?

(a) $X^5 - 4X + 22$

(b) $7x^4 + 25X^2 + 15X - 10$

(c) $2X^4 + 3X^3 + 3X^2 - 4$

(d) $5X^5 - 6X^4 + 12X^3 - 6$

Solution: The correct answer is (c): use Eisenstein's criteria to show that the other polynomials are irreducible. Note that

$$2X^4 + 3X^3 + 3X^2 - 4 = (2X^2 + X - 2)(X^2 + X + 2).$$

4. The polynomial $X^4 + 4X + 1$ is

(a) reducible in $\mathbb{Q}[X]$.

(b) irreducible in $\mathbb{Q}[X]$.

Solution: Substituting $X + 1$ for X, we obtain

$$(X + 1)^4 + 4(X + 1) + 1 = X^4 + 4X^3 + 6X^2 + 8X + 6.$$

By Eisenstein's criterion for $p = 2$ this is irreducible. Hence then correct answer is (b).

5. Which of the following statements is true?

(a) 7 is irreducible and prime in $\mathbb{Z}[\sqrt{-13}]$.

(b) 7 is irreducible but not prime in $\mathbb{Z}[\sqrt{-13}]$.

(c) 7 is neither irreducible nor prime in $\mathbb{Z}[\sqrt{-13}]$.

(d) 7 is prime but not irreducible in $\mathbb{Z}[\sqrt{-13}]$.

Solution: The correct answer is (b). Note that $7 \mid (1 + \sqrt{-13})(1 - \sqrt{-13}) = 2 \cdot 7$. To prove that 7 is irreducible, use the complex absolute value: from

$$|(a + b\sqrt{-13})(c + d\sqrt{-13})|^2 = (a^2 + 13b^2) \cdot (c^2 + 13d^2) = 49,$$

for $a, b, c, d \in \mathbb{Z}$, we obtain that either $a+b\sqrt{-13}$ is a unit, $c+d\sqrt{-13}$ is a unit or $a^2+13b^2 = 7$ (which is not solvable in $\mathbb{Z}$).